MULTIPLE SYMMETRIC INVARIANT NON TRIVIAL SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC VARIATIONAL SYSTEMS

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Abstract. In the present paper we prove a multiplicity result for a model quasi-linear elliptic system, coupled with the homogeneous Dirichlet boundary condition \((S_\lambda)\) on the unit ball, depending on a positive parameter \(\lambda\). By variational methods, we prove that for large values of \(\lambda\), the problem \((S_\lambda)\) has at least two non-zero symmetric invariant weak solutions.

Keywords: Elliptic system, Symmetrization, Ekeland’s variational principle, Spherical cap symmetrizations.

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1. Introduction

Consider the following quasi-linear, elliptic differential system coupled with the homogeneous Dirichlet boundary condition,
\[
\begin{align*}
-\Delta_p u &= \lambda F_u(x, u(x), v(x)) \quad \text{in } \Omega, \\
-\Delta_q v &= \lambda F_v(x, u(x), v(x)) \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where \(\lambda\) is a positive parameter and \(N > p, q > 1\), \(\Omega = B(0, 1) \subset \mathbb{R}^N\) is the unit ball, \(F \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R})\), \(F_z\) denotes the partial derivative of \(F\) with respect to \(z\), \(\Delta_\alpha\) is the \(\alpha\)-Laplacian operator, i.e., \(\Delta_\alpha = \text{div}(|\nabla u|^\alpha - 2 \nabla u)\). Systems of the type \((S_\lambda)\) have been the object of intensive investigations on bounded domains; We refer to the works of Boccardo and de Figueiredo [1], de Figueiredo [3], de Nápoli, Mariani [4] and Kristály, Rădulescu, Varga [6].

From the articles dealing with systems, we would like to highlight the paper of A. Kristály and I. Mezei, see [5], which studies a gradient-type system defined on a strip like domain, depending on two parameters, and proving a Ricceri-type three critical point result. While keeping some conditions from [5], we also aim to give a multiplicity theorem for our problem, three solutions, which are invariant under symmetrization.

As we already have pointed out, our aim is to examine the above problem in the point of view of symmetrizations, namely to prove a result which ensures the existence of symmetrically invariant solutions.

Despite of the fact, that symmetrizations don’t really occur in modelling real situations of the everyday life, they are very useful and highly applied topic in the
theory of partial differential equations. Many mathematicians worked and work in the study of symmetrizations, trying to describe some new phenomena. Here we would like to mention the work of Brock and Solynin, [2], J. Van Schaftingen [8], M. Squassina [7] who have proven many results among symmetrizations in the past few years, here we thinking of the symmetric minimax principle, Ekeland-, Borvein Preiss variational principles etc., which have opened many new ways to applications of this topic. In [2], Brock and Solynin proved that the Steiner symmetrization of a function can be approximated in $L^p(\mathbb{R}^n)$ by a sequence of very simple rearrangements which are called polarizations. Moreover, they introduced the concept of rearrangement and investigated some general properties.

The aforementioned problem is interesting not only from a mathematical point of view but also from its applicability in mathematical physics. The problem $(S_\lambda)$ is a generalization of the equation of the spring pendulum. A spring pendulum is a physical system where a piece of mass is connected to a spring so that the resulting motion contains elements of a simple pendulum motion as well as a spring motion. The equation of spring pendulum is the following:

$$
\begin{align*}
-\ddot{x}(t) &= \omega_0^2 x(t) \left(1 - \frac{l_0}{\sqrt{x(t)^2 + y(t)^2}}\right), \\
-\ddot{y}(t) &= \omega_0^2 y(t) \left(1 - \frac{l_0}{\sqrt{x(t)^2 + y(t)^2}}\right),
\end{align*}
$$

where $\omega_0 = \sqrt{\frac{g}{l_0}}$ and $l_0$ is the length of the spring at rest. A simple simulation shows how our numerical solutions be represented, and that the orbit of this kind of pendulum has a fractal-like shape. Such phenomena are often studied in chaos theory.

The problem $(S)$ can be treated as a variational problem, if we choose

$$
F(x, y) = \frac{\omega_0^2 (x^2 + y^2)}{2} - \omega_0^2 \cdot l_0 \sqrt{x^2 + y^2},
$$

then the energy functional associated to problem $S$ is defined by

$$
E(x, y) = \int_I (x')^2 + (y')^2 dt - \int_I F(x, y) dt,
$$

where $I \subset \mathbb{R}_+$. 

The main objective of our paper is to ensure the existence of symmetric invariant non-trivial solutions for the problem $(S_\lambda)$ where the natural functional framework is the Sobolev space $W^{1,p,q}_0(\Omega) = W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$.

In order to present our main result, we first recall that $(u, v) \in W^{1,p}_0 \times W^{1,q}_0$ is a weak solution to problem $(S_\lambda)$ if
for every \((w_1, w_2) \in W^{1,p}_0 \times W^{1,q}_0\).

In the sequel, we outline our approach and state the main result. We assume that the following hypotheses hold:

\(\mathcal{F}_1\) \(F : \Omega \times \mathbb{R}^2 \to \mathbb{R}\) is a continuous function, \((s, t) \mapsto F(x, s, t)\) is of \(C^1\) and \(F(x, 0, 0) = F(x, s, 0) = F(x, 0, t) = 0\) and \(F_s(x, s, t) \cdot s_+ + F_t(x, s, t) \cdot t_+ \leq 0\) for all \(x, s, t\), where \(\tau_+ = \min\{0, \tau\}\);

\(\mathcal{F}_2\) \(\lim_{(s,t) \to (0,0)} F(x, s, t) = 0\), uniformly for every \(x \in \Omega\);

\(\mathcal{F}_3\) \(\lim_{|s|+|t| \to +\infty} F(x, s, t) = 0\), uniformly for every \(x \in \Omega\);

\(\mathcal{F}_4\) There exists \((u_0, v_0) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)\) such that

\[\int_{\Omega} F(x, u_0(x), v_0(x)) dx > 0;\]

\(\mathcal{F}_5\) For \(F(x, s, t) = F(y, s, t)\) for each \(x, y \in \Omega\) with \(|x| = |y|\) and \(s, t \in \mathbb{R}\) and for \(x \in \Omega\) and \(a \leq b\) and \(c \leq d\)

\[F(x, a, c) + F(x, b, d) \geq F(x, a, d) + F(x, b, c);\]

\(\mathcal{F}_6\) For all \(x, s, t\) one has

\[F(x, s, t) \leq F(x, |s|, |t|).\]

Our main result reads as follows:

**Theorem 1.1.** Assume that \(p, q > 1\), and let \(\Omega \subset \mathbb{R}^N\) be the unit ball. Let \(F \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R})\) be a function which satisfies \((\mathcal{F}_1) - (\mathcal{F}_6)\). There exists a \(\lambda_0\) such that, for every \(\lambda > \lambda_0\) the problem \((S_{\lambda})\) has at least two weak solutions in \(W^{1,p,q}_0(\Omega)\), invariant by spherical cap symmetrization.

**Remark 1.1.** Let \(p = q = 2\), then the function \(F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) defined by \(F(x, s, t) = \|x\| \ln(1 + s^2 \cdot t^2)\) fulfills the hypotheses \((\mathcal{F}_1) - (\mathcal{F}_6)\), where \(\tau_+ = \max\{0, \tau\}\).

**Remark 1.2.** From \((\mathcal{F}_1)\) and \((\mathcal{F}_5)\) one can conclude the following inequality:

\[F(x, 0, 0) + F(x, s, t) \geq F(x, 0, t) + F(x, s, 0)\]

for \(t, s \geq 0\), therefore

\[F(x, s, t) \geq 0\]

for \(s, t \geq 0\).
Remark 1.3. If

\[ S_F = C_{\text{max}} \sup_{(s,t) \neq (0,0)} \frac{|sF_s(x,s,t) + tF_t(x,s,t)|}{|s|^p + |t|^q} < \infty, \]

then there exists a \( \lambda_F \) such that, for every \( 0 < \lambda \leq \lambda_F \) the problem \((S_{\lambda})\) has only the trivial solution. Indeed, a solution of \((S_{\lambda})\) is a pair \((u,v) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)\) such that

\[
\begin{aligned}
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla w_1 dx - \lambda \int_{\Omega} F_u(x, u(x), v(x)) w_1(x) dx &= 0 \\
\int_{\Omega} |\nabla v|^{q-2} \nabla v \nabla w_2 dx - \lambda \int_{\Omega} F_v(x, u(x), v(x)) w_2(x) dx &= 0,
\end{aligned}
\]

for all \( w_1 \in W^{1,p}_0(\Omega) \) and \( w_2 \in W^{1,q}_0(\Omega) \).

Choosing \( w_1 = u \) and \( w_2 = v \), we obtain that

\[
\|u\|_{1,p}^p + \|v\|_{1,q}^q = \lambda \int_{\Omega} (F_u(x, u(x), v(x))u + F_v(x, u(x), v(x))v) dx \leq \lambda \frac{S_F}{C_{\text{max}}} \int_{\Omega} |u|^p + |v|^q \leq \lambda \frac{S_F}{C_{\text{max}}} (C_p \|u\|_{1,p}^p + C_q \|v\|_{1,q}^q) \leq \lambda S_F (\|u\|_{1,p}^p + \|v\|_{1,q}^q),
\]

where, \( C_{\text{max}} = \max\{C_p, C_q\} \), therefore if \( \lambda < \frac{1}{S_F} \) then we necessarily have that \((u,v) = (0,0)\), which concludes the proof of this remark.

Remark 1.4. Note that if \((u,v)\) is a weak solution

\[
\begin{aligned}
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla u_- dx - \lambda \int_{\Omega} F_u(x, u(x), v(x)) u_- (x) dx &= 0 \\
\int_{\Omega} |\nabla v|^{q-2} \nabla v \nabla v_- dx - \lambda \int_{\Omega} F_v(x, u(x), v(x)) v_- (x) dx &= 0,
\end{aligned}
\]

follows that \( u_- = v_- = 0 \).

The proof of Theorem 1.1 is based on the symmetric version of the general minimax theorem and on a symmetric version of Ekeland’s variational principles. Therefore, in the next section we recall besides of the aforementioned results, some properties from the critical point theory(Palais-Smale sequence, Mountain pass theorem), while in Section 3, we prove our main result. As we already pointed out, the proof of this theorem is based on variational arguments. To see this, we consider the space \( W^{1,\alpha}_0(\Omega) \) endowed with the norm

\[
\|u\|_{1,\alpha} = \left( \int_{\Omega} |\nabla u|^\alpha \right)^{1/\alpha}, \quad \alpha \in \{p, q\};
\]
and for $\beta \in [\alpha, \alpha^*]$ we have the Sobolev embeddings $W^{1,\alpha}_0(\Omega) \hookrightarrow L^\beta(\Omega)$. The product space $W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$ is endowed with the norm $\|(u,v)\|_{1,p,q} = \|u\|_{1,p} + \|v\|_{1,q}$. We define the function $F : W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega) \to \mathbb{R}$ by
\[ F(u,v) = \int_\Omega F(x,u,v) \, dx \]
for $u \in W^{1,p}_0(\Omega), \ v \in W^{1,q}_0(\Omega)$. The energy functional associated to problem $(S_\lambda)$ is defined by
\[ (1.2) \quad A_\lambda(u,v) = \frac{1}{p} \|u\|_{1,p}^p + \frac{1}{q} \|v\|_{1,q}^q - \lambda \int_\Omega F(x,u,v) \, dx. \]

2. Preliminaries

This section is devoted to present some preparatory results. First, we recall the definition of the Palais-Smale condition, the Mountain Pass theorem, and we recall the definition of the spherical cap symmetrization and polarization, and some abstract results from symmetrization theory.

Definition 2.1. (Palais-Smale condition, [10], [6])

(a) A function $\varphi \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (shortly, $(PS)_c$-condition) if every sequence $\{u_n\} \subset X$ such that
\[ \lim_{n \to \infty} \varphi(u_n) = c \quad \text{and} \quad \lim_{n \to \infty} \|\varphi'(u_n)\| = 0, \]
possesses a convergent subsequence.

(b) A function $\varphi \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition (shortly, $(PS)$-condition) if it satisfies the Palais-Smale condition at every level $c \in \mathbb{R}$.

Theorem 2.1. (Mountain Pass Theorem, [10]) Let $X$ be a Banach space, $\varphi \in C^1(X, \mathbb{R}), e \in X$ and $r > 0$ be such that $\|e\| > r$ and
\[ b := \inf_{\|u\|=r} \varphi(u) > \varphi(0) \geq \varphi(e). \]
Then for each $\varepsilon > 0$ there exists $u \in X$ such that
(a) $c - 2\varepsilon \leq \varphi(u) \leq c + 2\varepsilon$
(b) $\|\varphi'(u)\| < 2\varepsilon$

where
\[ c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)) \]
and
\[ \Gamma := \{\gamma \in C([0,1]) : \gamma(0) = 0, \gamma(1) = e\}. \]

Moreover, if $\varphi$ satisfies the Palais-Smale condition at level $c$, then $c$ is a critical value of $\varphi$. 

**Definition 2.2** (Spherical cap symmetrization). Let \( P \in \partial B(0,1) \cap \mathbb{R}^N \), the spherical cap symmetrization of the set \( A \) with respect to \( P \) is the unique set \( A^* \) such that \( A^* \cap \{0\} = A \cap \{0\} \) and for any \( r \geq 0 \),
\[
A^* \cap \partial B(0,r) = B_g(rP,\rho) \cap \partial B(0,r) \text{ for some } \rho \geq 0,
\]
where \( H^{N-1} \) is the outer Hausdorff \((N-1)\)-dimensional measure and \( B_g(rP,\rho) \) denotes the geodesic ball on the sphere \( \partial B(0,r) \) of center \( rP \) and radius \( \rho \). By definition \( B_g(rP,0) = \emptyset \).

**Definition 2.3.** The spherical cap symmetrization of a function \( u : \Omega \to \mathbb{R} \) is the unique function \( u^* : \Omega^* \to \mathbb{R} \) such that, for all \( c \in \mathbb{R} \),
\[
\{u^* > c\} = \{u > c\}^*.
\]

**Definition 2.4** (Polarization). A subset \( H \) of \( \mathbb{R}^N \) is called a polarizer if it is a closed affine half-space of \( \mathbb{R}^N \), namely the set of points \( x \) which satisfy \( \alpha \cdot x \leq \beta \) for some \( \alpha \in \mathbb{R}^N \) and \( \beta \in \mathbb{R} \) with \( |\alpha| = 1 \). Given \( x \in \mathbb{R}^N \) and a polarizer \( H \), the reflection of \( x \) with respect to the boundary of \( H \) is denoted by \( x_H \). The polarization of a function \( u : \mathbb{R}^N \to \mathbb{R}^+ \) by a polarizer \( H \) is the function \( u^H : \mathbb{R}^N \to \mathbb{R}^+ \) defined by
\[
u^H(x) = \begin{cases} 
\max\{u(x),u(x_H)\}, & \text{if } x \in H \\
\min\{u(x),u(x_H)\}, & \text{if } x \in \mathbb{R}^N \setminus H. 
\end{cases}
\]

The polarization \( C^H \subset \mathbb{R}^N \) of a set \( C \subset \mathbb{R}^N \) is defined as the unique set which satisfies \( \chi_C u = (\chi_C)^H \), where \( \chi \) denotes the characteristic function. The polarization \( u^H \) of a positive function \( u \) defined on \( C \subset \mathbb{R}^N \) is the restriction to \( C^H \) of the polarization of the extension \( \tilde{u} : \mathbb{R}^N \to \mathbb{R}^+ \) of \( u \) by zero outside \( C \). The polarization of a function which may change sign is defined by \( u^H := |u|^H \), for any given polarizer \( H \).

2.1. **Abstract framework of symmetrizations.** Based on [8], consider the following abstract framework:

Let \( X \) and \( V \) be two real Banach spaces, with \( X \subset V \) and let \( S \subset X \). For the better understanding of the reader we present some crucial abstract symmetrization and polarization results of Van Schaftingen [8] and of Squassina [7]. Let us first introduce the following main assumption.

**Definition 2.5.** Let \( \mathcal{H}_s \) be a pathconnected topological space and denote by \( h : S \times \mathcal{H}_s \to S, (u,H) \mapsto u^H \), the polarization map. Let \( * : S \to V, u \mapsto u^* \), be any symmetrization map. Assume that the following properties hold.

1) The embeddings \( X \hookrightarrow V \) and \( V \hookrightarrow W \) are continuous;
2) \( h \) is continuous;
3) \((u^*)^H = (u^H)^* = u^* \) and \((u^H)^H = u^H \) for all \( u \in S \) and \( H \in \mathcal{H}_s \);
4) for all \( u \in S \) there exists a sequence \((H_m)_m \subset \mathcal{H}_*\) such that \( u^{H_1 \ldots H_m} \to u^* \) in \( V \);
5) \( \|u^H - v^H\|_V \leq \|u - v\|_V \) for all \( u, v \in S \) and \( H \in \mathcal{H}_* \).

Since there exists a map \( \Theta : (X, \| \cdot \|_V) \to (S, \| \cdot \|_V) \) which is Lipschitz continuous, with Lipschitz constant \( C_\Theta > 0 \), and such that \( \Theta|_S = Id|_S \), both maps \( h : S \times \mathcal{H}_* \to S \) and \( \star : S \to V \) can be extended to \( h : X \times \mathcal{H}_* \to S \) and \( \star : X \to V \) by setting \( u = (\Theta(u))^H \) and \( u^* = (\Theta(u))^* \) for every \( u \in X \) and \( H \in \mathcal{H}_* \).

The previous properties, in particular 4) and 5), and the definition of \( \Theta \) easily yield that

\[
\|u^H - v^H\|_V \leq C_\Theta \|u - v\|_V, \quad \|u^* - v^*\|_V \leq C_\Theta \|u - v\|_V
\]

for all \( u, v \in X \) and for all \( H \in \mathcal{H}_* \). We are now able to state the uniform approximation of symmetrization established by Van Schaftingen in [8].

**Proposition 2.1.** (Corollary 3.4 of [8]). For all \( \varepsilon > 0 \) there exists a continuous map \( T_\varepsilon : S \to S \) such that \( T_\varepsilon u \) is built via iterated polarization and \( \|T_\varepsilon u - u^*\|_V < \varepsilon \) for all \( u \in S \).

**Example 2.1.** (Spherical cap symmetrization with Dirichlet boundary condition) Let \( \Omega \) be now a ball or an annulus of \( \mathbb{R}^N \). Put \( X = W_0^{1,p}(\Omega), V = L^p(\Omega) \cap L^p(\Omega), \) with \( p^* = \frac{Np}{N-p} \). Denote by \( \star \) the spherical cap symmetrization and let \( \mathcal{H}_* \) be defined as above. Again the assumptions stated in Definition 2.5 are satisfied by Proposition 2.19, Theorem 2.20 and Proposition 2.20 of [8].

**Example 2.2.** (Schwarz symmetrization) Let again \( X = W_0^{1,p}(B), V = L^p(B) \cap L^p(B), \) with \( p^* = \frac{Np}{N-p} \), \( S = W_0^{1,p}(B), u^* = |u|^*, \) where \( \star \) denotes the Schwarz symmetrization and \( \mathcal{H}_* \) is defined as above for the Schwarz symmetrization, but \( u^H = |u|^H \). Then the assumptions stated in Definition 2.5 are satisfied again by Proposition 2.19, Theorem 2.20 and Proposition 2.20 of [8].

We recall three results which are crucial in our further investigations.

**Proposition 2.2.** Let \( H \in \mathcal{H}_* \). Suppose \( \Omega = \Omega^H \subset \mathbb{R}^N \), \( u, v : \Omega \to \mathbb{R} \) are measurable and nonnegative. If \( G : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ \) is a Borel measurable function such that \( G(x, s, t) = G(x, H(s), t) \) and if \( x \in \Omega \) and \( a \leq b \) and \( c \leq d \), \( G(x, a, c) + G(x, b, d) \geq G(x, a, d) + G(x, b, c) \) then

\[
\int_{\Omega} G(x, u, v)dx \leq \int_{\Omega} G(x, u^H, v^H).
\]

We recall a symmetric version of Ekeland’s variational principle, which appears in the paper [7] of Squassina.

**Theorem 2.2.** (Theorem 2.8 of [7]). Let \((X, V, \star, \mathcal{H}_*, S)\) satisfy the assumptions given in Definition 2.5. Denote by \( \kappa > 0 \) any constant with the property \( \|u\|_V \leq \kappa \|u\| \) for all \( u \in X \).
Assume that \( \Phi : X \to \mathbb{R} \cup \{ \infty \} \) is a proper lower semi-continuous functional bounded from below such that
\[
\Phi(u^H) \leq \Phi(u) \quad \text{for all } u \in S \text{ and } H \in \mathcal{H}_*.
\]
Moreover, assume that for all \( u \in \text{dom}(\Phi) \) there exists \( \xi \in S \) such that \( \Phi(\xi) \leq \Phi(u) \).

Then for all \( \varepsilon > 0 \) and \( \sigma > 0 \) there exists \( v \in X \) such that
\[
\begin{align*}
a) & \quad \|v - v^*\|_V < (\kappa(C_\Theta + 1) + 1)\varepsilon; \\
b) & \quad \Phi(w) \geq \Phi(v) - \sigma\|w - v\| \quad \text{for all } w \in X.
\end{align*}
\]
In addition, \( \Phi(v) \leq \Phi(u) \) and \( \|v - u\| \leq \varepsilon + \|T_u u - u\|, \) where \( u \in S \) is some element which satisfies \( \Phi(u) \leq \inf_X \Phi + \varepsilon\sigma \) and \( T_\varepsilon \) is the continuous map given in Proposition 2.1.

Now, we recall a symmetric version of Minimax Theorem, due to Van Schaftingen in [8].

In what follows \( X \) denotes a real Banach space, \( e \) a fixed element of \( X \setminus \{0\} \),
\[
\Gamma = \{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \quad \gamma(1) = e \},
\]
and \( \Phi \) a functional of class \( C^1(X) \) such that
\[
\tag{2.5} c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \Phi(\gamma(t)) > a = \max \{ \Phi(0), \Phi(e) \}.
\]

**Theorem 2.3** (Theorem 3.5 of [8]). Let \( (X, V, \star, H_*, S) \) satisfy the assumptions of Definition 2.5. Denote by \( \kappa > 0 \) any constant with the property \( \|u\|_V \leq \kappa\|u\| \) for all \( u \in X \). Let \( \Phi \in C^1(X) \) verify (2.5) and assume that \( \Phi(u^H) \leq \Phi(u) \) for all \( u \in S \) and \( H \in \mathcal{H}_* \). For every \( \varepsilon \in (0, (c - a)/2) \), \( \delta > 0 \) and \( \gamma \in \Gamma, \) with the properties
\[
\begin{align*}
i) & \quad \sup_{t \in [0, 1]} \Phi(\gamma(t)) \leq c + \varepsilon; \\
ii) & \quad \gamma([0, 1]) \subset S; \\
iii) & \quad \{ \gamma(0), \gamma(1) \}^H_0 = \{ \gamma(0), \gamma(1) \} \text{ for some } H_0 \in \mathcal{H}_*.
\end{align*}
\]
then there exists \( u_\varepsilon \in X \) such that
\[
\begin{align*}
a) & \quad c - 2\varepsilon \leq \Phi(u_\varepsilon) \leq c + 2\varepsilon; \\
b) & \quad \|u_\varepsilon - u_*\|_V \leq 2(2\kappa + 1)\delta; \\
c) & \quad \|\Phi'(u_\varepsilon)\|_{X^*} \leq 8\varepsilon/\delta.
\end{align*}
\]

**Remark 2.1.** In our case
\[
\Gamma = \{ \gamma \in C([0, 1], W_0^{1,p,q} \cap H_0^1) : \gamma(0) = (0, 0), \quad \gamma(1) = (e_u, e_v) \},
\]
where \((e_u, e_v) \in W_0^{1,p,q} \cap H_0^1\) is a fixed element different from \((0, 0)\).

### 3. Proof of Theorem 1.1

Before proving our main result, we prove that our functional \( \mathcal{A}_\lambda \) is coercive and satisfies the Palais- Smaile condition on \( W_0^{1,p,q}(\Omega) = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \).

**Lemma 3.1.** The functional \( \mathcal{A}_\lambda : W_0^{1,p,q}(\Omega) \to \mathbb{R} \) is coercive for every \( \lambda \geq 0 \).
Proof. Let us fix a \( \lambda \geq 0 \). Due to (\( F_2 \)) and (\( F_3 \)) one has that for every \( \varepsilon > 0 \), there exists \( \delta_1 = \delta_1(\varepsilon) > 0 \) and \( \delta_2 = \delta_2(\varepsilon) \) such that

(3.1) \( F(x, s, t) \leq \varepsilon \,(|s|^p + |t|^q) \), whenever \( |s| + |t| > \delta_1 \),

and

(3.2) \( F(x, s, t) \leq \varepsilon \,(|s|^p + |t|^q) \), whenever \( |s| + |t| < \delta_2 \).

From the definition of the function \( F \), there exists \( M_\varepsilon > 0 \) such that

\[ |F(x, s, t)| \leq M_\varepsilon \text{ whenever } |s| + |t| \in [\delta_2, \delta_1]. \]

Therefore, we obtain

\[ \mathcal{F}(u, v) = \int_\Omega F(x, u, v)dx = \int_{\{|u|_{1,p} + |v|_{1,q} > \delta_1\}} F(x, u, v)dx + \int_{\{|u|_{1,p} + |v|_{1,q} < \delta_2\}} F(x, u, v)dx + \int_{\{|u|_{1,p} + |v|_{1,q} \in [\delta_2, \delta_1]\}} F(x, u, v)dx \leq 2\varepsilon C_\alpha^p |u|_{1,p}^p + 2\varepsilon C_\alpha^q |v|_{1,q}^q + M_\varepsilon \omega_n, \]

where \( C_\alpha \) is the embedding constant in \( W_0^{1,\alpha}(\Omega) \hookrightarrow L^\alpha(\Omega), \alpha \in \{p, q\} \). Therefore

\[ \mathcal{A}_\lambda(u, v) \geq \left( \frac{1}{p} - 2\varepsilon \lambda C_\alpha^p \right) |u|_{1,p}^p + \left( \frac{1}{q} - 2\varepsilon \lambda C_\alpha^q \right) |v|_{1,q}^q - \lambda M_\varepsilon \omega_n \geq \]

\[ \geq \left( \frac{1}{p + q} - 2\varepsilon \lambda \max \{C_\alpha^p, C_\alpha^q\} \right) (|u|_{1,p}^p + |v|_{1,q}^q) - \lambda M_\varepsilon \omega_n, \]

where \( \lambda \max = \max \{C_\alpha^p, C_\alpha^q\} \).

In particular if \( 0 < \varepsilon < (2(p + q)\lambda \max)^{-1} \), then \( \mathcal{A}_\lambda \) is coercive, which concludes our proof. \( \square \)

Lemma 3.2. One has,

\[ \mathcal{A}_\lambda(u^H, v^H) \leq \mathcal{A}_\lambda(u, v). \]

Proof. First of all, we mention that, from \( F_6 \) one can conclude the following inequality

\[ \int_\Omega F(x, u(x), v(x))dx \leq \int_\Omega F(x, |u|(x), |v|(x))dx. \]

Based on the above inequality and on Remark 1.2 we can apply Proposition 2.2 for \( (u, v) \in W_0^{1,p,q}(\Omega)_+ \). Therefore, one has that \( \|\nabla u^H\|_{L^p} = \|\nabla u\|_{L^p}, \|\nabla v^H\|_{L^q} = \|\nabla v\|_{L^q} \) and \( u^H\|_{L^p} \leq \|u\|_{L^p} \leq \|u\|_{L^p} \).

On the other hand, due to \( F(x, s, t) = F(x_H, s, t) \) one has,

\[ \int_\Omega F(x, u(x), v(x))dx \leq \int_\Omega F(x, u^H(x), v^H(x))dx, \]
Remark 3.1. Using the Sobolev embeddings, \((F_1)\) and \((F_2)\) and \((F_3)\), one can prove in a standard way that \(F\) is of class \(C^1\), its differential being
\[
F(u, v)(w, y) = \int_\Omega |F_u(x, u, v)w + F_v(x, u, v)y|,
\]
for every \(u, w \in W_0^{1,p}(\Omega)\) and \(v, y \in W_0^{1,q}(\Omega)\).

Lemma 3.3. Let \(\lambda \geq 0\) be fixed and let \(\{(u_n, v_n)\}\) be a bounded sequence in \(W_0^{1,p,q}(\Omega)\) such that
\[
\|\mathcal{J}_\lambda'(u_n, v_n)\|_* \to 0
\]
as \(n \to \infty\). Then \(\{(u_n, v_n)\}\) contains a strongly convergent subsequence in \(W_0^{1,p,q}(\Omega)\).

Proof. Because \(W_0^{1,p,q}(\Omega)\) is a reflexive Banach space and \(\{(u_n, v_n)\}\) is a bounded sequence, we can assume that
\[
\begin{align*}
(3.3) & \quad (u_n, v_n) \to (u, v) \text{ weakly in } W_0^{1,p,q}, \\
(3.4) & \quad (u_n, v_n) \to (u, v) \text{ strongly in } L^p(\Omega) \times L^q(\Omega).
\end{align*}
\]
On the other hand, we have
\[
\mathcal{J}_\lambda'(u_n, v_n)(u - u_n, v - v_n) = \int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla u - \nabla u_n)
+ \int_\Omega |\nabla v_n|^{q-2}\nabla v_n(\nabla v - \nabla v_n) - \lambda \mathcal{F}'(u_n, v_n)(u - u_n, v - v_n)
\]
and
\[
\mathcal{J}_\lambda'(u, v)(u_n - u, v_n - v) = \int_\Omega |\nabla u|^{p-2}\nabla u(\nabla u_n - \nabla u)
+ \int_\Omega |\nabla v|^{q-2}\nabla v(\nabla v_n - \nabla v) - \lambda \mathcal{F}'(u, v)(u_n - u, v_n - v).
\]
Adding these two relations, one has
\[
a_n \overset{\text{not.}}{=} \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)
+ \int_\Omega (|\nabla v_n|^{q-2}\nabla v_n - |\nabla v|^{q-2}\nabla v)(\nabla v_n - \nabla v)
\right.
\]
\[
= -\mathcal{J}_\lambda'(u_n, v_n)(u - u_n, v - v_n) - \mathcal{J}_\lambda'(u, v)(u_n - u, v_n - v)
- \lambda \mathcal{F}'(u_n, v_n)(u - u_n, v - v_n) - \lambda \mathcal{F}'(u, v)(u_n - u, v_n - v).
\]
We can easily see that the last two terms tends to 0 as \(n \to \infty\). Due to (3.3), the second terms tends to 0, while the inequality
\[
|\mathcal{J}_\lambda'(u_n, v_n)(u - u_n, v - v_n)| \leq \|\mathcal{J}_\lambda'(u_n, v_n)\|_*(u - u_n, v - v_n)\|_{p,q}
\]
and the assumption implies that the first term tends to 0 too. Thus,
\[ \lim_{n \to \infty} a_n = 0. \]

From the well-known inequality
\[ |t - s|^{\alpha} \leq \begin{cases} (|t|^{\alpha-2} - |s|^{\alpha-2})(t - s), & \text{if } \alpha \geq 2, \\ (\alpha/2(|t|^{\alpha} + |s|^{\alpha})(2-\alpha)/2, & \text{if } 1 < \alpha < 2, \end{cases} \]
for all \( t, s \in \mathbb{R}^N \), and (3.5), we conclude that
\[ \lim_{n \to \infty} \int_{\Omega} (|\nabla u_n - \nabla u|^p + |\nabla v_n - \nabla v|^q) = 0, \]
hence, the sequence \( \{u_n, v_n\} \) converges strongly to \((u, v)\) in \( W^{1,p,q}_0 \).

**Proof of Theorem 1.1** The proof is divided into two steps. In the first one we prove that the critical point obtained by the symmetric version of Ekeland principle is symmetric invariant, while in the second step we prove that the critical point obtained by the symmetric version of the minimax theorem is invariant under symmetrization.

**(Step 1.)** For the minimizing sequence \((u_n, v_n)\) we consider the following sequence
\[ \varepsilon_n = \begin{cases} \mathcal{A}(u_n, v_n) - d, & \text{if } \mathcal{A}(u_n, v_n) - d > 0 \\ \frac{1}{n}, & \text{if } \mathcal{A}(u_n, v_n) - d = 0, \end{cases} \]
where \( d = \inf \mathcal{A} \). We have that, \( \mathcal{A}(u_n, v_n) \leq \varepsilon_n \) and \( \varepsilon_n \to 0 \) as \( n \to \infty \). Applying Theorem 2.2, yields that there exists a sequence \( \{(a_n, b_n)\} \subset W^{1,p,q}_0(\Omega) \) such that:

(a) \( \mathcal{A}(a_n, b_n) \leq \mathcal{A}(u_n, v_n) \);

(b) \( \|a_n, b_n\|_{LR \times L^q} \to 0 \);

Since \( \mathcal{A} \) satisfies the (PS) condition, up to a subsequence \( (a_n, b_n) \to (a, b), (a, b) \in W^{1,p,q}_0(\Omega) \). From the fact that \( (a_n, b_n) \to (a, b) \) and \( (a_n^*, b_n^*) \to (a^*, b^*) \) we can conclude that
\[ \|(a, b) - (a^*, b^*)\|_{LR \times L^q} \leq \]
\[ \leq \|a_n, b_n\|_{LR \times L^q} + \|a_n^*, b_n^*\|_{LR \times L^q} + \|a_n^* - a_n^*, b_n^* - b_n^*\|_{LR \times L^q}. \]

Therefore from the above outcomes one has that \( (a, b) = (a^*, b^*) \). Which conclude the proof of the first step.

**(Step 2.)**
\[ \mathcal{A}((u_0, v_0)) = \frac{1}{p} \|u_0\|_{L^p}^p + \frac{1}{q} \|v_0\|_{L^q}^q - \lambda \int_{\Omega} F(x, u_0(x), v_0(x))dx = A - \lambda B, \]
where \( A = \|u_0\|_{L^p}^p + \|v_0\|_{L^q}^q > 0 \), and \( B = \int_{\Omega} F(x, u_0(x), v_0(x))dx > 0 \). Consequently, there exists \( \lambda_0 > 0 \) such that for every \( \lambda > \lambda_0 \), we have that \( h(\lambda) = A - \lambda B < 0 \),
therefore
\[ \mathcal{A}_\lambda((u_0, v_0)) = \frac{1}{p} \|u_0\|_{1,p}^p + \frac{1}{q} \|v_0\|_{1,q}^q - \lambda \int_\Omega F(x, u_0(x), v_0(x)) \, dx < 0. \]

In fact, we may choose,
\[ \lambda_0 = \inf \left\{ \frac{\frac{1}{p} \|u\|_{1,p}^p + \frac{1}{q} \|v\|_{1,q}^q}{\mathcal{F}(u, v)} : u \in W_0^{1,p}(\Omega), v \in W_0^{1,q}(\Omega), \mathcal{F}(u, v) > 0 \right\}. \]

Now, fix \( \lambda > \lambda_0 \). First, relations \((\mathcal{F}_2)\) and \((\mathcal{F}_3)\) imply that for every \( \epsilon > 0 \) there exists \( \delta_1 = \delta_1(\epsilon), \delta_2 = \delta_2(\epsilon) \) such that for every \((s, t) \in \mathbb{R}^2\) with \(|s| + |t| \in (0, \delta_1) \cup (\delta_2, +\infty)\), one has
\[ 0 \leq \frac{F(x, s, t)}{|s|^p + |t|^q} < \epsilon. \]

Fix \( \gamma \in \left(1, \min \left\{ \frac{p^*}{p}, \frac{q^*}{q} \right\} \right) \). Note that the continuous function \((s, t) \mapsto \frac{F(x, s, t)}{|s|^p + |t|^q}\) is bounded on the set \(\{(s, t) \in \mathbb{R}^2 : |s| + |t| \in [\delta_1, \delta_2]\}\). Therefore there exists \(m_\epsilon > 0\), such that
\[ F(x, s, t) \leq \frac{\epsilon}{\max \{pC_p^p, qC_q^q\}} (|s|^p + |t|^q) + \frac{m_\epsilon}{(\max \{p, q\})^\gamma} (|s|^p + |t|^q), \]
for all \((s, t) \in \mathbb{R}^2\).

Therefore, for each \((u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)\) we get
\[ \mathcal{F}(u, v) = \int_\Omega F(x, u, v) \leq \]
\[ \leq \frac{\epsilon}{\max \{pC_p^p, qC_q^q\}} \int_\Omega |u|^p + |v|^q \, dx + \frac{m_\epsilon}{(\max \{p, q\})^\gamma} \int_\Omega (|u|^p + |v|^q) \leq \]
\[ \leq \frac{\epsilon}{\max \{pC_p^p, qC_q^q\}} (\|u\|_{L^p}^p + \|v\|_{L^q}^q) + \frac{m_\epsilon}{(\max \{p, q\})^\gamma} (\|u\|_{L^p}^p + \|v\|_{L^q}^q) \leq \]
\[ \leq \frac{\epsilon}{\max \{pC_p^p, qC_q^q\}} (C_p^p \|u\|_{1,p}^p + C_q^q \|v\|_{1,q}^q) + \frac{m_\epsilon}{(\max \{p, q\})^\gamma} (C_p^p \|u\|_{1,p}^p + C_q^q \|v\|_{1,q}^q) \leq \]
\[ \leq \epsilon \left( \frac{1}{p} \|u\|_{1,p}^p + \frac{1}{q} \|v\|_{1,q}^q \right) + m_\epsilon C_{\gamma} \left[ \left( \frac{1}{p} \|u\|_{1,p}^p \right)^\gamma + \left( \frac{1}{q} \|v\|_{1,q}^q \right)^\gamma \right] \leq \]
\[ \leq \epsilon \left( \frac{1}{p} \|u\|_{1,p}^p + \frac{1}{q} \|v\|_{1,q}^q \right) + m_\epsilon C_{\gamma} \left( \frac{1}{p} \|u\|_{1,p}^p + \frac{1}{q} \|v\|_{1,q}^q \right)^\gamma, \]
where \( C_{\gamma} = \max \{C_{\gamma}^p, C_{\gamma}^q\} \) and in the last inequality we used the fact that the function
\((s, t) \mapsto (s^\gamma + t^\gamma)^{\frac{1}{\gamma}}\), \(s, t \geq 0\),
is decreasing. Consequently
\[ \mathcal{A}_\lambda(u, v) \geq \]
\[ \geq (1 - \epsilon) \left( \frac{1}{p} \|u\|_{1,p}^p + \frac{1}{q} \|v\|_{1,q}^q \right) - \lambda m_\epsilon C_{\gamma} \left( \frac{1}{p} \|u\|_{1,p}^p + \frac{1}{q} \|v\|_{1,q}^q \right)^\gamma \]
there exists a critical point \((u,v)\), which means that

\[
\frac{1}{p+q} \left( \|u\|^p_{1,p} + \|v\|^q_{1,q} \right) - \lambda m_c C_\gamma \max \left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( \|u\|^p_{1,p} + \|v\|^q_{1,q} \right)^\gamma = \\
\left( \frac{1}{p+q} - \lambda m_c C_\gamma \max \left\{ \frac{1}{p}, \frac{1}{q} \right\} \right) \left( \|u\|^p_{1,p} + \|v\|^q_{1,q} \right)^{\gamma-1} \left( \|u\|^p_{1,p} + \|v\|^q_{1,q} \right).
\]

Now let \(0 < \rho < 1\), and \(\|(u,v)\| = \|u\|_{1,p} + \|v\|_{1,q}\). Then we have

\[
\left( \frac{\rho}{2} \right)^{\max\{p,q\}} \leq \|u\|^p_{1,p} + \|v\|^q_{1,q} \leq \rho.
\]

Therefore for \(\rho\) small enough \(\mathcal{A}_\lambda(u,v) > 0\).

Since

\[
\inf_{\|(u,v)\| = \rho} \mathcal{A}_\lambda(u,v) = \mathcal{A}_\lambda(0,0) > \mathcal{A}_\lambda(u_0, v_0)
\]

and \(\mathcal{A}_\lambda\) satisfies the Palais-Smale condition, we are in the position to apply the mountain pass theorem with

\[
\Gamma = \{ \gamma \in C([0,1], W^{1,p,q}_0) : \gamma(0) = (0,0), \quad \gamma(1) = (e_u, e_v) \},
\]

which means that \(c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{A}_\lambda(\gamma(t))\) is a critical value of \(\mathcal{A}_\lambda\), therefore there exists a critical point \((u,v)\) such that \(\mathcal{A}_\lambda(u,v) = c\).

From the definition of \(c\),

\[
\sup_{t \in [0,1]} \mathcal{A}_\lambda(\gamma(t)) \leq c + \frac{1}{n^2}.
\]

From the above inequality and from the first step, \(\mathcal{A}_\lambda\) has a global minimum which is invariant by spherical symmetrization, and \((0,0)\) is also invariant by spherical symmetrization (therefore the assumption iii) in Theorem 2.3 is fulfilled) we are in the position to apply Theorem 2.3 with

\[
\Gamma = \{ \gamma \in C([0,1], W^{1,p,q}_0) : \gamma(0) = (0,0), \quad \gamma(1) = (e_u, e_v) \},
\]

where \((e_u, e_v)\) is the global minimum of \(\mathcal{A}_\lambda\). Choosing \(\varepsilon = \frac{1}{n^2}\), and \(\delta = \frac{1}{n}\), yields that there exist \((u_n, v_n) \in W^{1,p,q}_0(\Omega)\) such that

(a) \(|\mathcal{A}_\lambda(u_n, v_n) - c| \leq \frac{2}{n^2}\);

(b) \(\|(u_n, v_n) - (u^*_n, v^*_n)\|_{L^p \times L^q} \leq 2(2K + 1)\frac{1}{n}\);

(c) \(\|\mathcal{A}''_\lambda(u_n, v_n)\| \leq \frac{2}{n}\).

Since \(\mathcal{A}_\lambda\) satisfies the \((PS)\) condition by Lemma 3.3 there exist a subsequence \((u_{n_k}, v_{n_k})\) of \((u_n, v_n)\) and \((u,v) \in W^{1,p,q}_0(\Omega)\) such that \((u_{n_k}, v_{n_k}) \to (u,v)\) as \(k \to \infty\). Hence the inequalities (a), (b), (c) imply that \(\mathcal{A}_\lambda(u,v) = c\) and \(\mathcal{A}''_\lambda(u,v) = 0\) and \((u,v) = (u^*, v^*)\) since

\[
\|(u,v) - (u^*, v^*)\|_{L^p \times L^q} \leq \|(u,v) - (u_{n_k}, v_{n_k})\|_{L^p \times L^q} + \|(u_{n_k}, v_{n_k}) - (u^*_n, v^*_n)\|_{L^p \times L^q} + \|(u^*_n, v^*_n) - (u^*, v^*)\|_{L^p \times L^q} \to 0,
\]

which concludes our proof. \(\square\)
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