EXPLORATORY PROJECT 2011, PN-II-ID-PCE-2011-3-0241

SYMMETRIES IN ELLIPTIC PROBLEMS: EUCLIDEAN AND NON-EUCLIDEAN TECHNIQUES

MEMBERS:

- **Project Manager:** Prof. univ. dr. Alexandru Kristály
- **Senior Researcher:** Prof. univ. dr. Csaba Varga
- **Young Researchers:** Lector dr. Csaba Farkas, Lector dr. Ildikó Mezei

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1In the period of 07 February - 31 December 2013, I. Mezei (young researcher) was on a maternity leave.
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1 Project presentation

Variational methods based on symmetrization technics are important tools in the theory of elliptic problems; outstanding researchers work nowadays within this topic.

The primary goal of the present project is to study symmetrization phenomena in nonlinear elliptic problems within non-Euclidean frameworks (on Riemann-Finsler manifolds, on Carnot and Heisenberg groups, on (Sierpinski-type) fractals or even on metric measure spaces). First, there are investigated elliptic PDEs which involve the Finsler-Laplace operator associated with asymmetric Minkowski norms modeling for instance the Matsumoto mountain slope metric or various Randers-type norms coming from Mathematical Physics. Quantitative anisotropic isoperimetric inequalities, improved anisotropic Hardy-Rellich inequalities, and the Wulff shape of minimizers for sharp optimal constants in important integral inequalities are considered by employing critical point theory and symmetrization arguments for asymmetric Minkowski norms. Second, the members of the team are interested to identify those group actions which are hidden in non-Euclidean phenomena by solving elliptic PDEs via the principle of symmetric criticality.

Under the supervision of the Project Manager and prof. Csaba Varga, there was initiated a Scientific Seminar in the direction of the present project: symmetrization techniques on Euclidean and non-Euclidean spaces (Schwartz and Steiner symmetrizations, Pólya-Szegő-type inequality, Hardy-Littlewood inequality, Cacciopoli principle), polarization, variational arguments, Lions’ concentration compactness, etc. These seminars are taken by the project members on every Thursday, 9 a.m. in Cluj-Napoca.

As a direct consequence of this intensive activity, the project members obtained spectacular results during the research period, publishing in top journals as Math. Annalen, Calc. Var. and PDEs, J. Math. Pures Appl., J. Funct. Anal., J. Geom. Anal., Potential Analysis, J. Optim. Theory Appl. and Nonlinear Analysis. In the sequel we shall present the papers obtained during the research period.

2 Accepted/published papers in 2011-2016

2.1 Reported papers in 2011-2012


Cumulative AIS (2011-2012): 5.62
Cumulative IF (2011-2012): 3.597

Both AIS and IF are taken from the 2016 lists provided by UEFISCDI.
2.2 Reported papers in 2012-2013


Cumulative AIS (2012-2013): 8.665
Cumulative IF (2012-2013): 5.502

2.3 Reported papers in 2013-2014


Cumulative IF (2013-2014): 8.161
2.4 Reported papers in 2014-2015


2.5 Reported papers in 2015-2016


Cumulative AIS (2015-2016): 10.015
Cumulative IF (2015-2016): 7.961
### 2.6 Scientometric table

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### 3 Short presentation of accepted/published papers in the period 2011-2016

#### 3.1 Period 2011-2012


In this paper the authors prove a Lions-type compactness embedding result for symmetric unbounded domains of the Heisenberg group. It is proved that the natural group action on the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ having the operation $(z, t) \circ (z', t') = (z + z', t + t' + 2\text{Im}(z, z'))$ is provided by the unitary group $U(n) \times \{1\}$ and its appropriate subgroups which will be used to construct subspaces with specific symmetry and compactness properties in the Folland-Steins horizontal Sobolev space, where $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $t \in \mathbb{R}$ and $\langle z, z' \rangle = \sum_{j=1}^{n} z_j \overline{z'_j}$. By using this algebraic operation, one can generate the space of symmetric spaces on the horizontal Sobolev space where the horizontal derivative is given by means of the vector fields

\[
X_j^1 = \partial_{x_j} + 2y_j \partial_t,
X_j^2 = \partial_{y_j} - 2x_j \partial_t,
T = \partial_t.
\]

As an application, the authors study the multiplicity of solutions for a singular subelliptic problem by exploiting a technique of solving the Rubik-cube applied to subgroups of $U(n) \times \{1\}$. *This paper is in conformity with the objective C2.II.3.*


The paper focuses on the existence of nontrivial solutions of a nonlinear eigenvalue problem depending on a real parameter $\lambda$ under Robin boundary conditions in unbounded domains, with (possibly noncompact) smooth boundary. Denoting by $\lambda_1$ the first eigenvalue of the underlying Robin eigenvalue problem, the authors proved the existence of (weak) solutions, with different methods, according to the case $\lambda \geq \lambda_1$ or $\lambda < \lambda_1$. In the first part of the paper they showed the existence of a nontrivial solution for all $\lambda \in \mathbb{R}$ for the problem under Ambrosetti-Rabinowitz-type conditions on the nonlinearities involved in the model. In detail, the authors applied the mountain-pass theorem of Ambrosetti and Rabinowitz if $\lambda < \lambda_1$, while they used mini-max methods and linking structures over cones, as in the paper by Degiovanni and by Degiovanni and Lancelotti, if $\lambda \geq \lambda_1$. In the latter part of the paper they do not require any longer the Ambrosetti-Rabinowitz condition at $\infty$, but the so-called Szulkin-Weth conditions, and they obtain the same result for all $\lambda \in \mathbb{R}$. More precisely, using the Nehari-manifold method for $C^1$ functionals developed by Szulkin and Weth, the authors proved existence of ground states, multiple solutions, and least-energy sign-changing solutions, whenever $\lambda < \lambda_1$. On the other hand, in the case $\lambda \geq \lambda_1$ they established the existence of solutions again by linking methods. *This paper is in conformity with the objective C2.II.1.*

In this paper the authors studied the following quasilinear equation coupled with a homogeneous Neumann boundary condition:

\[
\begin{cases}
-\Delta_p u + |u|^{p-2} u = \lambda \alpha(x,y) f(u) & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(N_\lambda)

where \( \Omega = \omega \times \mathbb{R}^{N-m} \), \( \omega \subset \mathbb{R}^m \) being bounded and open with smooth boundary, \( p > N \), \( N - m \geq 2 \), \( \Delta_p \) is the \( p \)-Laplacian operator, \( \lambda \) is a positive parameter, \( \alpha \in L^\infty(\Omega) \) is a non-zero potential with compact support, \( n \) is the outward normal vector, and \( f : [0, \infty) \to \mathbb{R} \) is a continuous function with \( f(0) = 0 \). They first established a compact embedding result and then they prove a multiplicity result for the problem \((N_\lambda)\) on strip-like domains. Using variational methods, they proved that for large values of \( \lambda \), problem \((N_\lambda)\) has at least two non-zero weak solutions, while there exists at least a \( \tilde{\lambda} > 0 \) such that problem \((\tilde{N}_\lambda)\) has at least three non-zero weak solutions. These solutions show symmetry properties with respect to certain group actions. *This paper is in conformity with the objective C2.II.1.*


In this paper the authors studied the following problem:

\[
\begin{cases}
-\Delta_p u = \lambda f(x,u) + \mu |u|^{q-2} u & \text{in } \Omega; \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega; \\
\|u\|_q = 1,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with a smooth boundary \( \partial \Omega \), \( 1 < p < q < p^* \) (being \( p^* \) the Sobolev critical exponent) and \( \Delta_p \) denotes the \( p \)-Laplace operator. By means of critical point theory on manifolds, the authors established the existence of one, two or three solutions for the above problem. *This paper is in conformity with the objective C2.II.2.*

### 3.2 Period 2012-2013


Motivated by Nash equilibrium problems on ‘curved’ strategy sets, the concept of Nash-Stampacchia equilibrium points is introduced via variational inequalities on Riemannian manifolds. Characterizations, existence, and stability of Nash-Stampacchia equilibria are studied when the strategy sets are compact/noncompact geodesic convex subsets of Hadamard manifolds, exploiting two well-known geometrical features of these spaces both involving the metric projection map. These properties actually characterize the non-positivity of the sectional curvature of complete and simply connected Riemannian spaces, delimiting the Hadamard manifolds as the optimal geometrical framework of Nash-Stampacchia equilibrium problems. This analytical approach exploits various elements from set-valued and variational analysis, dynamical systems, and non-smooth calculus on Riemannian manifolds. Examples are presented on the Poincaré upper-plane model and on the open convex cone of symmetric positive definite matrices endowed with the trace-type Killing form. *This paper is in conformity with the objective C2.II.2.*

The authors prove that if a metric measure space satisfies the volume doubling condition and the Caffarelli-Kohn-Nirenberg inequality with the same exponent \( n \geq 3 \), then it has exactly the \( n \)-dimensional volume growth. As an application, if an \( n \)-dimensional Finsler manifold of non-negative \( n \)-Ricci curvature satisfies the Caffarelli-Kohn-Nirenberg inequality with the sharp constant, then its flag curvature is identically zero. In the particular case of Berwald spaces, such a space is necessarily isometric to a Minkowski space. *This paper is in conformity with the objectives C2.I.1 & C2.I.3.*


There are treated nonlinear elliptic and parameter-depending problems, defined on the Sierpinski gasket, a highly non-smooth fractal set. Even if the structure of this fractal differs considerably from that of (open) domains of Euclidean spaces, the paper emphasizes that PDEs defined on it may be studied (as in the Euclidean case) by means of certain variational methods. Using such methods, and some recent abstract multiplicity theorems by B. Ricceri, there are proved several results concerning the existence of multiple solutions of three-parameter Dirichlet problems defined on the Sierpinski gasket. *This paper is in conformity with the objective C2.II.1.*


The existence of multiple solutions for a constrained eigenvalue problem on a compact Riemannian manifold without boundary is proved. In the framework of variational analysis, the authors used an Arcoya-Carmona type three critical points result. As an application an Emden-Fowler type equation is given. *This paper is in conformity with the objective C2.II.2.*

### 3.3 Period 2013-2014


In this paper the author studied a sharp Sobolev interpolation inequality on Finsler manifolds. It is proved that:

(a) Minkowski spaces support the sharp Sobolev interpolation inequality;

(b) Any complete Berwald space with non-negative Ricci curvature which supports the sharp Sobolev interpolation inequality is isometric to a Minkowski space.

In this way, one can assert that Minkowski spaces represent the optimal framework for the Sobolev interpolation inequality on a large class of Finsler manifolds. The proofs are based on properties of the Finsler-Laplace operator and on the Finslerian Bishop-Gromov volume comparison theorem. *This paper is in conformity with the objective C2.II.2.*


In this paper, two Morrey-Sobolev inequalities (with support-bound and \( L^1 \)-bound, respectively) are investigated on complete Riemannian manifolds with their sharp constants in \( \mathbb{R}^n \). The following results are proved in both cases:

(a) If \((M, g)\) has *non-negative Ricci curvature*, \((M, g)\) supports the sharp Morrey-Sobolev inequalities if and only if \((M, g)\) is isometric to \((\mathbb{R}^n, e)\).
If \((M,g)\) is a Cartan-Hadamard manifold which verifies the \(n\)-dimensional Cartan-Hadamard conjecture, sharp Morrey-Sobolev inequalities hold on \((M,g)\). Moreover, extremals exist if and only if \((M,g)\) is isometric to the standard Euclidean space \((\mathbb{R}^n,e)\).

In these arguments there are exploited symmetrization techniques on Riemannian manifolds and properties of the Beltrami-Laplace operator depending on the curvature of the space. This paper is in conformity with the objective C2.II.2.


In this paper the authors consider a nonlinear eigenvalue problem under Robin boundary conditions in a domain with (possibly noncompact) smooth boundary, namely,

\[
\begin{aligned}
-\text{div}(a(x)|\nabla u|^p-2\nabla u) &= \lambda f(x)|u|^{p-2}u + g(x,u) \quad \text{in } \Omega, \\
a(x)|\nabla u|^p-2\nabla u \cdot \nu + b(x)|u|^{p-2}u &= h(x,u) \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \(\Omega\) is a domain with the boundary \(\partial \Omega\), \(\lambda \in \mathbb{R}\) is a parameter and \(g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}\), \(h : \partial \Omega \times \mathbb{R} \to \mathbb{R}\) are Carathéodory functions satisfying some growth conditions. Using Morse theory and a cohomological local splitting theorem, the authors prove the existence of a nontrivial weak solution for all (real) values of the eigenvalue parameter. This result is new even in the semilinear case \(p=2\) and complements some recent results obtained in G. Autuori, P. Pucci, Cs. Varga [Adv. Differential Equations, 2013]. This paper is in conformity with the objective C2.II.1.


Using a recent result of B. Ricceri [A class of nonlinear eigenvalue problems with four solutions, J. Nonlinear Convex Anal. 11(2010), 503–511], the authors studied the existence and multiplicity of solutions for an elliptic system defined on the Sierpinski gasket depending on a parameter. One can prove that there exists a parameter such that the studied problem has at least three nontrivial distinct solutions. This paper is in conformity with the objective C2.II.1.


In this paper, the authors study Sobolev spaces on the \(n\)-dimensional unit ball \(B^n(1)\) endowed with a parameter-depending Finsler metric \(F_a\), \(a \in [0,1]\), which interpolates between the Klein metric \((a=0)\) and Funk metric \((a=1)\), respectively. It is showed that the standard Sobolev space defined on the Finsler manifold \((B^n(1),F_a)\) is a vector space if and only if \(a \in [0,1)\). Furthermore, by exploiting variational arguments, the authors provide non-existence and existence results for sublinear elliptic problems on \((B^n(1),F_a)\) involving the Finsler-Laplace operator whenever \(a \in [0,1)\).

We emphasize that this paper is the first, as far as we know, concerning the application of the principle of symmetric criticality within a non-Euclidean context. Therefore, this paper is in full conformity with the objective C2.II.2.


In this paper the authors deal with a quasilinear elliptic equation involving a critical nonlinearity and a lower order perturbation. Under very general assumptions on the perturbation it is proved the existence of a solution. The approach is based on the direct methods of calculus of variations. This paper is in conformity with the objective C2.II.2.

In the present paper the authors prove a multiplicity result for a model quasi-linear elliptic system, coupled with the homogeneous Dirichlet boundary condition on the unit ball, depending on a positive parameter. By variational methods, it is proved that for large values of the parameter, the studied problem has at least two non-zero symmetric invariant weak solutions. *This paper is in conformity with the objective C2.II.1.*


In the present paper the authors prove some multiplicity results for hemivariational inequalities defined on the unit ball or on the whole space. By variational methods, it is proved that the solutions of these inequalities are invariant by spherical cap symmetrization, the main tools being the symmetric version of Ekeland’s variational principle proved by M. Squassina [Symmetry in variational principles and applications, J. Lond. Math. Soc. 85(2012), 323-348] and a nonsmooth version of the symmetric minimax principle due to J. Van Schaftingen [Symmetrization and minimax principles, Commun. Contemp. Math., 7(2005), 463-481]. *This paper is in conformity with the objective C2.II.1.*

3.4 Period 2014-2015


In this paper the members of the research group studied a singular elliptic problem of Poisson type on Finsler-Hadamard manifolds. In order to elaborate the results, the authors used several arguments from the calculus of variations and Finsler geometry.

In the first part of the paper the authors studied a nontrivial problem from the theory of Sobolev spaces: they proved that the structure of the Sobolev space over a Finsler manifold \((M, F)\) depends on the so-called reversibility constant

\[
 r_F = \sup_{x \in M} \sup_{y \in T_x M \setminus \{0\}} \frac{F(x, y)}{F(x, -y)}. \]

More precisely, if \(r_F < +\infty\) then the Sobolev space on \((M, F)\) is reflexive and while \(r_F = +\infty\), the Sobolev space has not even a vector space structure. This result answers in some sense the question of the structure of Sobolev spaces on generic Finsler manifolds formulated in the paper of S. Ohta and K.-T. Sturm [Heat flow on Finsler manifolds. *Comm. Pure Appl. Math.* 62 (2009), no. 10, 1386–1433].

In the second part of the paper the authors established some rigidity results considering the following problem: what kind of geometric/analytic properties can be established for the singular Poisson equation? By using various techniques from Finsler geometry -as comparison arguments and symmetrization methods- some rigidity results have been established. Such a result can read as follows: if the solution of the Poisson equation has the profile of the Minkowskian radial problem for every ball then the Finsler manifold is isometric to the Euclidean space.

**Remark:** The primordial motivation of the present project is concentrated around such kind of problems, i.e., to describe the geometric impact of certain nonlinear phenomena from the theory of elliptic problems by means of euclidean and non-euclidean techniques. *Therefore, this paper is in full conformity with the objectives C2.I.2, C2.I.3, C2.II.2.*

In this paper the authors solve a problem raised by C. Gutiérrez and A. Montanari [Maximum and comparison principles for convex functions on the Heisenberg group. *Comm. Partial Differential Equations* 29 (2004), no. 9-10, 1305–1334] about comparison principles for H-convex functions on subdomains of Heisenberg groups. The approach is based on the notion of the sub-Riemannian horizontal normal mapping and uses degree theory for set-valued maps. The statement of the comparison principle combined with a Harnack inequality is applied to prove the Aleksandrov-type maximum principle, describing the correct boundary behavior of continuous H-convex functions vanishing at the boundary of horizontally bounded subdomains of Heisenberg groups. This result answers a question by N. Garofalo and F. Tournier [New properties of convex functions in the Heisenberg group. *Trans. Amer. Math. Soc.* 358 (2006), no. 5, 2011–2055]. The sharpness of the results are illustrated by examples. *This paper is in conformity with the objective C2.II.3.*


In this paper the authors studied a quasilinear elliptic problem coupled with Dirichlet boundary conditions. The existence of infinitely many solutions for such type of problems has been intensively studied under different assumptions on the nonlinearity. On one hand, it is well known that symmetry assumptions on the nonlinearity can yield infinitely many solutions for such kind of problem. On the other hand, one can guarantee the existence of infinitely many solutions when the nonlinearity exhibits a suitable oscillatory behavior either at zero or at infinity. With direct methods of calculus of variations the authors proposed a new set of assumptions ensuring the existence of infinitely many solutions for such kind of problem. *This paper is in conformity with the objective C2.I.3.*


Using the symmetric version of Ekeland’s variational principle of M. Squassina [Symmetry in variational principles and applications, *J. London Math. Soc.* 85 (2012), 323–348] and the Mountain Pass theorem due to J. Van Schaftingen [Symmetrization and minimax principles, *Comm. Contemp. Math.* 7 (2005), 463–481], the authors establish the existence of two nontrivial (weak) solutions of abstract eigenvalue problems depending on a real parameter. First, they present a slight modification of the Pucci and Serrin three critical points theorem. Then they prove a symmetric version of this result which is applies to the same abstract eigenvalue problem in order to show the existence of three different symmetric solutions. As a consequence of the main results, the authors prove the existence of two nontrivial nonnegative solutions which are invariant by $k$–spherical cap symmetrization, $1 = k < N$, of quasilinear elliptic Dirichlet problems in either a ball of $\mathbb{R}^N$ or an annulus of $\mathbb{R}^N$, both centered at 0. *This paper is in conformity with the objective C2.I.3.*


The authors show that the spectrum of a nonhomogeneous Baouendi-Grushin type operator subject with a homogeneous Dirichlet boundary condition is exactly the interval $(0, \infty)$. This is in sharp contrast with the situation when one deals with the classical Baouendi-Grushin operator when the spectrum is an increasing and unbounded sequence of positive real numbers. The proofs rely on a symmetric mountain-pass argument due to Kajikiya. In addition, they show that for each eigenvalue there exists a sequence of eigenfunctions converging to zero. *This paper is in conformity with the objective C2.I.3.*
By using Ekeland's variational principle the authors obtained a critical point theorem of Schechter type for extrema of a functional in an annular conical domain of a Banach space. The result can be seen as a variational analogue of Krasnoselskii's fixed point theorem in cones and can be applied for the existence, localization and multiplicity of the positive solutions of variational problems. The result is then applied to $p$–Laplace equations, where the geometric condition on the boundary of the annular conical domain is established via a weak Harnack type inequality given in terms of the energetic norm. This method can be applied also to other homogeneous operators in order to obtain existence, multiplicity or infinitely many solutions for certain classes of quasilinear equations. \textit{This paper is in conformity with the objective C2.I.3 and C2.II.1.}

3.5 \textbf{Period 2015-2016}


In this paper the authors deal with improved Rellich inequalities on Finsler-Hadamard manifolds with vanishing mean covariation where the remainder terms are expressed by means of the flag curvature. By exploiting various arguments from Finsler geometry the authors show that more weighty curvature implies more powerful improvements. The sharpness of the involved constants are also studied. \textit{This paper is in conformity with the objective C2.I.2.}


In this paper the authors study nonlinear Schrödinger-Maxwell systems on $n$–dimensional Hadamard manifolds, $3 \leq n \leq 5$. The main difficulty resides in the lack of compactness of such manifolds which is recovered by exploring suitable isometric actions. By combining variational arguments, some existence, uniqueness and multiplicity of isometry-invariant weak solutions are established for the Schrödinger-Maxwell system depending on the behavior of the nonlinear term. \textit{This paper is in conformity with the objective C2.II.2.}


Let $(M, d, m)$ be a metric measure space which satisfies the Lott-Sturm-Villani curvature-dimension condition $CD(K, n)$ for some $K \geq 0$ and $n \geq 2$, and a lower $n$–density assumption at some point of $M$. The author proves that if $(M, d, m)$ supports the Gagliardo-Nirenberg inequality or any of its limit cases ($L^p$--logarithmic Sobolev inequality or Faber-Krahn-type inequality), then a global non-collapsing $n$–dimensional volume growth holds, i.e., there exists a universal constant $C_0 > 0$ such that $m(B_x(\rho)) \geq C_0 \rho^n$ for all $x \in M$ and $\rho \geq 0$, where $B_x(\rho) = \{y \in M : d(x, y) < \rho\}$. Due to the quantitative character of the volume growth estimate, the author establishes several rigidity results on Riemannian manifolds with non-negative Ricci curvature supporting Gagliardo-Nirenberg inequalities by exploring a quantitative Perelman-type homotopy construction developed by Munn (J. Geom. Anal., 2010). Further rigidity results are also presented on some reversible Finsler manifolds. \textit{This paper is in conformity with the objective C2.I.1 and C2.I.3.}

The authors present geodesic versions of the Borell-Brascamp-Lieb, Brunn-Minkowski and entropy inequalities on the Heisenberg group $\mathbb{H}^n$. Their arguments use Riemannian approximation of $\mathbb{H}^n$ combined with optimal mass-transportation techniques. This paper is in conformity with the objective C2.II.3.


In the present paper the authors extend some recent results of R. Filippucci, P. Pucci and Cs. Varga to continuous functionals. As an application they prove the existence of at least three different solutions of a quasilinear eigenvalue problem for every $\lambda$ in some interval whose solutions are invariants w.r.t. Schwarz symmetrization. This paper is in conformity with the objective C2.II.1.


The authors of the present paper generalize M. Schechter’s theory to locally Lipschitz functionals defined on a closed ball of a real reflexive Banach space with strictly convex dual. Schechter’s original theory is applicable only to $C^1$-functionals defined on a Hilbert space, therefore their generalization is twofold. This paper is in conformity with the objective C2.I.3

- R. Precup, P. Pucci, Cs. Varga, A three critical point result in a bounded domain of a Banach space and applications, accepted, August 2016. *Differential and Integral Equations*, AIS: 1.376; IF: 0.81.

Using the bounded mountain pass lemma and the Ekeland variational principle the authors of the present paper prove a bounded version of the Pucci-Serrin three critical points result in the intersection of a ball with a wedge in a Banach space. The localization constraints are overcome by boundary and invariance conditions. The result is applied to obtain multiple positive solutions for some semilinear problems. This paper is in conformity with the objective C2.II.1.

4 Mobilities, collaborations and presentations

4.1 Period 2011-2012

A. Kristály, Project Manager:


- Nonlinear Difference and Differential Equations and their Applications, 4-6 October 2012, Rousse, Bulgaria. Title of presentation: *Anisotropic problems in the presence of asymmetric Minkowski norms*.


Cs. Varga, Senior Researcher:


- Nonlinear Difference and Differential Equations and their Applications, 4-6 October 2012, Russe, Bulgaria. Title of presentation: *Existence theorems for quasilinear elliptic problems in unbounded domains.*


Cs. Farkas, Young researcher:


- Nonlinear Difference and Differential Equations and their Applications, 4-6 October 2012, Russe, Bulgaria. Title of presentation: *Some multiplicity results on strip like domains.*

I. Mezei, Young Researcher:


4.2 Period 2012-2013

A. Kristály, Project Manager:


Cs. Varga, Senior Researcher:

- Öbuda University, Budapest, Hungary, 19-21 November 2013. Title of presentation: *Multiple solutions of Dirichlet problems and systems of gradient type on the Sierpinski gasket.*

Cs. Farkas, Young researcher:


- Workshop "Geometric Analysis on Riemannian and singular metric spaces", 30 September-4 October, 2013, Como, Italy, participant.

- Öbuda University, Budapest, Hungary, 19-21 November 2013. Title of presentation: *New conditions for the existence of infinitely many solutions for a quasilinear problem.*

Workshop & external collaborators

- In the period 14-15 March 2013, A. Kristály and Cs. Varga organized a workshop entitled "*Advances in Differential Equations: symmetrizations and related topics*", at Babeş-Bolyai University, Cluj-Napoca, Mathematicum. Participants: B. Breckner (Cluj), N. Costea (IMAR, București), Cs. Farkas (member of the project), J. Kolumbán, jr. (Cluj), A. Kristály (project manager), M. Mihăilescu (Univ. of Craiova), A. Molnár (Cluj), Gh. Moroșanu (Central European University, Budapest), Sz. Nagy (Cluj), A. Petrușel (Cluj). R. Precup (Cluj), Cs. Varga (member of the project).

External collaborators:

- Dr. Francesca Faraci, Universita di Catania, Catania, Italy, 9-14 April 2013. Title of the talk presented during the visit: "*On a bifurcation problem with an asymptotically linear nonlinearity*".

- Dr. Jess Garcia-Falset, University of Valencia, Spain, 22-26 October 2013.

4.3 Period 2013-2014

A. Kristály, Project Manager:


- ICMC - Summer Meeting on Differential Equations (2014 Chapter), Sao Paulo, Brasil, 3-7 February 2014. Title of presentation: *Caffarelli-Kohn-Nirenberg inequalities on metric measure spaces: symmetrization and rigidity.*
  http://summer.icmc.usp.br/summers/summer14/

- ALEL2014-International Conference on Optimization, Seville, Spain, 5-7 June 2014. Invited speaker. Title of presentation: *Nash-type equilibria on Riemannian manifolds: the effect of curvature.*
  http://www.imus.us.es/ALEL14/speakers.html

  https://www.aimsciences.org/AIMS-Conference/

- Scientific collaboration (with prof. F. Faraci): Universitatea din Catania, Catania, Italy, 14-25 August 2014.
• Scientific collaboration (with prof. Ph. G. Ciarlet): City University of Hong Kong, Hong Kong, China, 21 September - 20 October 2014.

Cs. Varga, Senior Researcher:

• Eötvös Loránd University, Budapest, Hungary, 22 September 2014. Title of presentation: Nonexistence and existence results for Finsler–Laplace equations with–singular weight.

• Scientific collaboration (with prof. L. Simon): Eötvös Loránd University, Budapest, Hungary, 2-11 November 2014.

Cs. Farkas, Young researcher:

• Scientific collaboration (with prof. S. Tersian): University of Russe, Bulgaria, 24-31 January 2014.


• Scientific collaboration (with prof. F. Faraci): Universita di Catania, Catania, Italy, 14-25 August 2014.

External collaborators:

• Professor Zoltán M. Balogh, Universität Bern, Switzerland, visited A. Kristály in the period of 27-28 May 2014.

• Professor László Székelyhidi, Jr., Universität Leipzig, Germany, visited Cs. Farkas, A. Kristály and Cs. Varga in the period 19-22 November 2014. Title of his presentation: The Euler equations: anomalous dissipation and non-uniqueness.

4.4 Period 2014-2015

A. Kristály, Project Manager:


• Scientific collaboration (with Prof. Ph. G. Ciarlet): City University of Hong Kong, Hong Kong, China, 1-31 March 2015.


• Scientific collaboration (with Prof. F. Faraci): Universita di Catania, Catania, Italy, 19-30 August 2015.


• Central European University, Budapest, Hungary, 14 September 2015. Title of presentation: What kind of geometric information is encoded into best Sobolev constants?

• Scientific collaboration (with Prof. Z. M. Balogh): Universität Bern, Bern, Switzerland, 11-31 October 2015.

Cs. Varga, Senior Researcher:

• Eötvös Loránd University, Budapest, Hungary, 27 April 2015. Title of presentation: A Schechter type critical point result in annular conical domains of a Banach space and applications.


• Eötvös Loránd University, Budapest, Hungary, 14 September 2015. Title of presentation: Localization of positive critical points in Banach spaces and applications.

• University of Pécs, Pécs, Hungary, 2-6 November 2015. Title of presentation: Location results in critical point theory and applications.

Cs. Farkas, Young Researcher:


• Scientific collaboration (with prof. F. Faraci): Universita di Catania, Catania, Italy, 29 September -28 October 2015. Title of presentation: Elliptic equations on Finsler manifolds.

Ildikó Mezei, Young Researcher:

• Research Seminar (Group of Analysis), Babeş-Bolyai University, Cluj-Napoca, Faculty of Mathematics and Informatics, 26 March 2015. Title of presentation: Symmetrizations and multiplicity theorems.
• Scientific collaboration (with Prof. F. Faraci): Universita di Catania, Catania, Italy, 19-30 August 2015.

• 5th MATINFO International Conference, 2-5 September 2015, Sapientia Hungarian University of Transylvania, Tg.-Mureș, Romania. Title of presentation: *Schwarz symmetric solutions for E-differentiable functionals.*
  

### 4.5 Period 2015-2016

**A. Kristály**, Project Manager:


• Scientific collaboration (with prof. A. Cabada): University of Santiago de Compostela, Santiago de Compostela, Spain, 6-12 September 2016. Title of presentation: *Gagliardo-Nirenberg inequalities on metric measure spaces: volume non-collapsing and rigidities.*

**Cs. Varga**, Senior Researcher:

• Workshop Geometry and PDEs, West University, Timișoara, Romania, 10-11 June 2016. Title of presentation: *Singular Poisson equations on Finsler-Hadamard Manifolds.*
  
  [http://www.math.uvt.ro/invatamant/cicburidoctoratmatematicaanunturi/Program.pdf](http://www.math.uvt.ro/invatamant/cicburidoctoratmatematicaanunturi/Program.pdf)

• Scientific collaboration: Eötvös Loránd University, Budapest, Hungary, 1-7 May 2016.

**Cs. Farkas**, Young Researcher:

• Scientific collaboration (with prof. A. Cabada): University of Santiago de Compostela, Santiago de Compostela, Spain, 6-12 September 2016.


**Ildikó Mezei**, Young Researcher:


**External collaborators:**

• **Dr. Francesca Faraci**, Universita di Catania, Catania, Italy, 9-14 May 2016. Title of the talk presented during the visit: *"Existence and multiplicity results for a semi-linear elliptic problem with a singular term".*

• **Professor Zoltán Balogh**, University of Bern, Bern, Switzerland, 2-6 June 2016.
5 Submitted papers


6 Final notes

- Mr. Csaba Farkas defended his PhD Thesis on 31 October 2014, the title being "Symmetrization methods in the study of elliptic problems". The aim of the thesis is to present new existence and multiplicity results in the study of sublinear elliptic problems in different contexts, combining techniques both from calculus of variations and the theory of symmetrization.

- As one can see from the list of submitted papers (§5), the members of the research project are working continuously in the description of new phenomena which are related to the objectives of the project, elaborating methods and technics based on the theory of symmetrization, Riemann-Finsler geometry, fractals, and Heisenberg groups.

- The formulated scientific objectives for the whole research period have been fully achieved.

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Project Manager,

26 September 2016

Prof. univ. dr. A. Kristály