NEW CONDITIONS FOR THE EXISTENCE OF INFINITELY MANY SOLUTIONS FOR A QUASI-LINEAR PROBLEM

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Abstract In this paper we study a quasi-linear elliptic problem coupled with Dirichlet boundary conditions. We propose a new set of assumptions ensuring the existence of infinitely many solutions.

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1. Introduction

In this paper we deal with the following Dirichlet problem

$$
\begin{cases}
-\Delta_p u = h(x)f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

(\mathcal{P})

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary, $p > 1$, $\Delta_p$ is the $p$-Laplacian operator, i.e, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, $f: \mathbb{R} \to \mathbb{R}$ is a continuous function, $h: \Omega \to \mathbb{R}$ is a bounded non-negative function. The existence of infinitely many solutions for such types of problem has been intensively studied under different assumptions on the nonlinearity. Since the pioneering work of Ambrosetti and Rabinowitz \cite{1} it has been well known that symmetry assumptions on $f$ can yield infinitely many solutions for (\mathcal{P}). The same result also holds when the symmetry of the energy is broken by the presence of a perturbation (see, for example, the contributions of Hirano and Zou \cite{8} and of Tehrani \cite{16}).

A different approach to the problem was proposed by Ricceri \cite{12,13} by means of suitable variational arguments, by Omari and Zanolin \cite{10,11} who exploited subsupersolution methods, and by Saint Raymond in \cite{15} (see also \cite{2,6,9}), where a sequence of local minima of the energy functional in suitable convex sets is exhibited. All these contributions prove the existence of infinitely many solutions when the nonlinearity exhibits
a suitable oscillatory behaviour either at zero or at infinity. It is worth mentioning that the oscillation of \( f \) in itself is not enough to guarantee multiple solutions: indeed, de Figueiredo [5] proved the uniqueness of positive solutions of (\( \mathcal{P} \)) when \( f(u) = \lambda \sin u \).

In the aforementioned results, if \( F \) is the primitive of \( f \), the oscillation of \( f \) is ensured by the assumption

\[
-\infty < \liminf_{t \to \ell} \frac{F(t)}{|t|^p} < \limsup_{t \to \ell} \frac{F(t)}{|t|^p} \leq +\infty,
\]

where \( \ell = \pm \infty, 0^\pm \).

In this paper we develop a variant of a recent existence and localization theorem by Ricceri [14] in order to prove the existence of infinitely many solutions for (\( \mathcal{P} \)) under new conditions on the nonlinearity. First of all, our result can be applied when

\[
\lim_{t \to \ell} \frac{F(t)}{|t|^p} \in \mathbb{R}
\]

(see Example 4.1). This is not the sole novelty of our result. We point out that the result of [14] is a consequence of the variational methods contained in [12]. Applicability of Ricceri’s variational principle (see [12]) in the framework of infinitely many weak solutions for quasi-linear problems is only known in low dimension, i.e. for \( p > N \) (see, for example, [3, 4, 7]).

We give a positive contribution also when \( p \leq N \), as our paper seems to provide the very first example in this direction (see Example 4.2). In conclusion, our result represents a step forward in the research of new conditions for finding infinitely many weak solutions for (\( \mathcal{P} \)).

The structure of the paper is the following: in \( \S \) 2 we introduce the preliminaries, while \( \S \) 3 contains our main result. To conclude the paper, \( \S \) 4 is devoted to some examples.

2. Notation and preliminaries

Let us introduce the notation we will use.

If \( N \geq p \), \( A \) denotes the class of continuous functions \( f: \mathbb{R} \to \mathbb{R} \) such that

\[
\sup_{t \in \mathbb{R}} \frac{|f(t)|}{1 + |t|^\gamma} < +\infty,
\]

where \( 0 < \gamma < p^* - 1 \) if \( p < N \) (here \( p^* = pN/(N - p) \)) and \( 0 < \gamma < +\infty \) if \( p = N \), while if \( N < p \), \( A \) is the class of continuous functions \( f: \mathbb{R} \to \mathbb{R} \). Denote by \( F \) the primitive of \( f \), i.e.

\[
F(t) = \int_0^t f(s) \, ds.
\]

Problem (\( \mathcal{P} \)) has a variational structure. More precisely, if \( f \in A \), the functional \( \mathcal{E}: W^{1,p}_0(\Omega) \to \mathbb{R} \) defined by

\[
\mathcal{E}(u) = \frac{1}{p} \|u\|^p - \int_\Omega h(x)F(u(x)) \, dx,
\]

where

\[
-\infty < \liminf_{t \to \ell} \frac{F(t)}{|t|^p} < \limsup_{t \to \ell} \frac{F(t)}{|t|^p} \leq +\infty,
\]
where $\| \cdot \|$ is the classical norm in the Sobolev space $W^{1,p}_0(\Omega)$, i.e.

$$ \|u\| = \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}, $$

is of class $C^1$ in $W^{1,p}_0(\Omega)$, and its critical points are precisely the weak solutions of problem $(\mathcal{P})$.

Denote by $B(0, \varrho)$ the open ball in $W^{1,p}_0(\Omega)$ centred at zero and of radius $\varrho$. Also, let $0 \leq a < b \leq +\infty$. For a pair of functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$, if $\lambda \in [a, b]$, we denote by $M(\varphi, \psi, \lambda)$ the set of all global minima of the function $\lambda \psi - \varphi$ or the empty set according to whether $\lambda < +\infty$ or $\lambda = +\infty$. We adopt the conventions $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$. We also put

$$ \alpha(\varphi, \psi, b) = \max \left\{ \inf_{\mathbb{R}} \psi, \sup_{M(\varphi, \psi, b)} \psi \right\}, $$

and

$$ \beta(\varphi, \psi, a) = \min \left\{ \sup_{\mathbb{R}} \psi, \inf_{M(\varphi, \psi, a)} \psi \right\}. $$

See Figures 1 and 2 for the geometrical meaning of $\alpha$ and $\beta$.

Furthermore, let

$$ q \in \begin{cases} [0, p^*) & \text{if } N > p, \\ [0, +\infty) & \text{if } N \leq p, \end{cases} $$
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\[ \psi_y \psi_x \psi_\lambda \phi = a \]

\[ H(\phi, \psi, a) := \{ \text{global minima of } \psi - \lambda \phi \} \]

Figure 2. Geometrical meaning of \( \beta \).

\[ c_q = \sup_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |u(x)|^q \, dx}{(\int_\Omega |\nabla u(x)|^p \, dx)^{q/p}}. \]

Denote by \( \mathcal{F}_q \) the family of all lower semi-continuous functions \( \psi : \mathbb{R} \to \mathbb{R} \), with \( \sup_{\mathbb{R}} \psi > 0 \), such that

\[ \inf_{t \in \mathbb{R}} \frac{\psi(t)}{1 + |t|^q} > -\infty \]

and

\[ \gamma_\psi := \sup_{t \in \mathbb{R} \setminus \{0\}} \frac{\psi(t)}{|t|^q} < +\infty. \]

In [14], Ricceri proved the following result.

**Theorem 2.1 (Ricceri [14]).** Let \( f \in A \) and let \( h \in L^\infty(\Omega) \setminus \{0\} \) with \( h \geq 0 \). Moreover, assume that there exists \( \psi \in \mathcal{F}_q \) such that, for each \( \lambda \in [a,b] \), the function \( \lambda \psi - F \) is coercive and has a unique global minimum in \( \mathbb{R} \). Finally, suppose that there exists a number \( r > 0 \) satisfying

\[ \alpha(F, \psi, b) < r < \beta(F, \psi, a) \]

and

\[ \sup_{\psi^{-1}(r)} F < \frac{r^{p/q}}{p(\gamma_\psi \text{ ess sup}_{\Omega} hc_q)^{p/q}(\int_{\Omega} h(x) \, dx)^{(q-p)/q}}. \]  \( \tag{2.1} \)
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Put
\[ d = \left( \frac{\int_{\Omega} h(x) \, dx}{\gamma \psi \text{ess sup}_{\Omega} h c} \right)^{p/q}. \]

Then problem \((\mathcal{P})\) has a weak solution \(u\) with \(\|u\| < r^{p/q}\). More precisely, \(u\) is a global minimum of the restriction of \(\mathcal{E}\) to \(B(0, dr^{p/q})\).

**Remark 2.2.** Notice that, from [14, Proposition A], for any \(r \in ]a, b[\) there exists \(\lambda_r \in ]a, b[\) such that the unique global minimum of \(\lambda_r \psi - F\) lies in \(\psi^{-1}(r)\). In particular, \(\psi^{-1}(r) \neq \emptyset\).

### 3. Main results
#### 3.1. The zero case

We deal with the existence of infinitely many solutions tending to zero in the norm of \(W^{1,p}_0(\Omega)\).

**Theorem 3.1.** Let \(f \in A\) and let \(h \in L^\infty(\Omega) \setminus \{0\}\) with \(h \geq 0\). Assume that there exists \(\psi \in \mathcal{F}_q\) such that, for each \(\lambda \in ]a, b[\), the function \(\lambda \psi - F\) is coercive and has a unique global minimum in \(\mathbb{R}\). Finally, suppose that
\[
\alpha(F, \psi, b) \leq 0 < \beta(F, \psi, a),
\]
and that 0 is not a local minimum of \(\mathcal{E}\).

Under such hypotheses, problem \((\mathcal{P})\) has a sequence of non-zero weak solutions \(\{u_n\}\) with
\[
\lim_{n \to \infty} \|u_n\| = 0.
\]
Also, \(\mathcal{E}(u_n) < 0\) for any \(n \in \mathbb{N}\), and \(\{\mathcal{E}(u_n)\}\) is increasing.

**Proof.** From condition (3.1) we can fix a decreasing sequence \(\{r_n\}\) of positive numbers such that
\[
\lim_{n} r_n = 0
\]
and
\[
\sup_{\psi^{-1}(r_n)} F < \frac{r_n^{p/q}}{p(\gamma \psi \text{ess sup}_{\Omega} h c)^{p/q}(\int_{\Omega} h(x) \, dx)^{(q-p)/q}} \quad \forall n \in \mathbb{N}. \tag{3.2}
\]
Fix \(k_1 \in \mathbb{N}\) such that
\[
\alpha(F, \psi, b) < r_{k_1} < \beta(F, \psi, a).
\]
From Theorem 2.1 we deduce the existence of \(u_1 \in W^{1,p}_0(\Omega)\), which is a weak solution of \((\mathcal{P})\) and satisfies
\[
\|u_1\|^p < dr_{k_1}^{p/q}.
\]
Since \( u_1 \) is a local minimum of \( E \), \( u_1 \neq 0 \). Now, choose \( k_2 \in \mathbb{N} \), \( k_2 > k_1 \), such that
\[
r_{k_2} \leq d^{-q/p} \| u_1 \|^q.
\]
As
\[
\alpha(F, \psi, b) < r_{k_2} < r_{k_1} < \beta(F, \psi, a),
\]
there exists a weak solution of \((P)\) \( u_2 \neq 0 \) such that \( \| u_2 \| < \frac{d}{r_{k_2}} \). Hence, by the choice of \( r_{k_2} \), we also deduce that \( \| u_2 \| < \| u_1 \| \). Therefore, we can construct a sequence \( \{ u_n \} \) of non-zero solutions of \((P)\) such that, for every \( n \in \mathbb{N} \),
\[
\| u_n \| < d \frac{p}{q} r_{k_n}
\]
and
\[
\| u_n \| < \| u_{n-1} \|.
\]
In particular, \( \lim_{n} \| u_n \| = 0 \) and \( u_n \) are pairwise distinct. It is also clear that \( E(u_n) < 0 \) for all \( n \in \mathbb{N} \) and that \( \{ E(u_n) \} \) is increasing. \( \square \)

**Remark 3.2.** From the proof, it is clear that Theorem 3.1 holds with (3.2) in place of (3.1).

We now propose some sufficient conditions ensuring that 0 is not a local minimum of \( E \).

**Lemma 3.3.** Assume one of the following conditions:

\[
(i_{0+}) - \infty < \lim_{t \to 0^+} \frac{F(t)}{t^p} \leq \lim_{t \to 0^+} \frac{F(t)}{t^p} = +\infty,
\]

\[
(i_{0-}) - \infty < \lim_{t \to 0^+} \frac{F(t)}{|t|^p} \leq \lim_{t \to 0^+} \frac{F(t)}{|t|^p} = +\infty.
\]

Then 0 is not a local minimum of \( E \).

**Proof.** Let us prove \((i_{0+})\). Since \( h \in L^\infty(\Omega) \setminus \{0\}, h \geq 0 \), it is possible to find \( x_0 \in \Omega \) and \( \theta > 0 \) such that \( B(x_0, \theta) \subset \Omega \), and \( h(x) > (1/p)\|h\|_\infty \) almost everywhere in \( B(x_0, \theta) \). For every \( s \in \mathbb{R}, s \neq 0 \), define
\[
w_s(x) = \begin{cases} 
0 & \text{in } \Omega \setminus B(x_0, \theta), \\
\frac{2s}{\theta} (\theta - |x - x_0|) & \text{in } B(x_0, \theta) \setminus B\left(x_0, \frac{\theta}{2}\right), \\
s & \text{in } B\left(x_0, \frac{\theta}{2}\right).
\end{cases}
\]

Then \( w_s \in W_{0}^{1,p}(\Omega) \) and from direct calculation we have that
\[
\| w_s \|_p = \left(\frac{\theta}{2}\right)^{N-p} (2^N - 1)|\omega_N| |s|^p \quad (3.3)
\]
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(where \( \omega_N \) is the measure of the unit ball in \( \mathbb{R}^N \)). From the left-hand inequality of our assumption, we deduce the existence of \( l > 0 \) and \( \delta > 0 \) such that

\[
F(t) > -lt^p \quad \text{for } 0 < t < \delta.
\]

Fix \( L \) such that

\[
L > (2^N - 1) \left[ lp + \frac{2p}{\|h\|_\infty \theta^p} \right].
\]

(3.4)

Using the right-hand inequality of \((i_{0^+})\), there exists a sequence \( \{s_n\} \subset \mathbb{R}^+, s_n \to 0^+ \), such that

\[
F(s_n) > L s_n^p \quad \text{for every } n \in \mathbb{N}.
\]

Then we have, for \( n \in \mathbb{N} \),

\[
\mathcal{E}(w_{s_n}) = \frac{1}{p} \left( \theta \right)^{2-N-p} \left( 2^N - 1 \right) \omega_N s_n^p - \int_{B(x_0, \theta/2)} h(x) F(s_n) \]

\[
- \int_{B(x_0, \theta) \backslash B(x_0, \theta/2)} h(x) F(w_{s_n})
\]

\[
\leq \frac{1}{p} \left( \theta \right)^{2-N-p} \left( 2^N - 1 \right) \omega_N s_n^p - \frac{1}{p} \|h\|_\infty \left( \theta \right)^N \omega_N L s_n^p
\]

\[
+ \|h\|_\infty l \left( \theta \right)^N \left( 2^N - 1 \right) \omega_N s_n^p
\]

\[
= \left( \theta \right)^N \omega_N s_n^p \left[ \frac{1}{p} \left( \theta \right)^{-p} \left( 2^N - 1 \right) + l \|h\|_\infty (2^N - 1) - \frac{1}{p} L \|h\|_\infty \right]
\]

< 0

as it follows from (3.4). Therefore,

\[
\mathcal{E}(w_{s_n}) < 0 = \mathcal{E}(0).
\]

Remark 3.4. From the proof of Lemma 3.3, we can weaken condition \((i_{0^+})\) by assuming that

\[
(j_{0^+}) \liminf_{t \to 0^+} \frac{F(t)}{t^p} > -\infty \quad \text{and} \quad \limsup_{t \to 0^+} \frac{F(t)}{t^p} > (2^N - 1) \left[ p \liminf_{t \to 0^+} \frac{F(t)}{t^p} + \frac{2p}{\|h\|_\infty \theta^p} \right].
\]

Analogously, we can replace \((i_{0^-})\) with

\[
(j_{0^-}) \liminf_{t \to 0^-} \frac{F(t)}{t^p} > -\infty \quad \text{and} \quad \limsup_{t \to 0^-} \frac{F(t)}{t^p} > (2^N - 1) \left[ p \liminf_{t \to 0^-} \frac{F(t)}{t^p} + \frac{2p}{\|h\|_\infty \theta^p} \right].
\]
Remark 3.5. The conclusion of Lemma 3.3 is also valid under the conditions
\[
(k_0^0) \text{ ess inf } h > 0 \text{ and } \liminf_{t \to 0^+} \frac{F(t)}{t^p} > \frac{1}{pc_p \text{ ess inf}_\Omega h},
\]
\[
(k_0^-) \text{ ess inf } h > 0 \text{ and } \liminf_{t \to 0^-} \frac{F(t)}{|t|^p} > \frac{1}{pc_p \text{ ess inf}_\Omega h}.
\]

Proof. See [14]. □

Corollary 3.6 easily follows from Theorem 3.1.

Corollary 3.6. Let \( f \in A \) and \( \psi \in F \) such that, for each \( \lambda \in ]a,b[ \), the function \( \lambda \psi - F \) is coercive, has a unique global minimum in \( \mathbb{R} \) and one of the conditions \((i_0^0)\), \((i_0^-)\) hold. Finally, suppose that
\[
\alpha(F,\psi,b) \leq 0 < \beta(F,\psi,a)
\]
and
\[
\liminf_{t \to 0^+} \frac{\sup_{\psi^{-1}(c)} F}{t^{p/q}} < +\infty. \tag{3.5}
\]

Under such hypotheses, there exists \( \mu^* > 0 \) such that for every \( \mu \in ]0,\mu^*[ \), the problem
\[
-\Delta_p u = \mu f(u) \quad \text{in } \Omega,
\]
\[
u = 0 \quad \text{on } \partial \Omega,
\]
has a sequence of non-zero weak solutions \( \{u_n\} \) with
\[
\lim_{n \to \infty} ||u_n|| = 0.
\]

When \( \liminf_{t \to 0^+} F(t)/t^p = 0 \) in condition \((j_0^0)\) we can easily obtain (3.1) of Theorem 3.1 as in the following theorem.

Theorem 3.7. Let \( f \in A \) with \( f(0) \geq 0 \), and let \( h \in L^\infty(\Omega) \setminus \{0\} \) with \( h \geq 0 \). Assume for each \( \lambda \in ]0,+\infty[ \) that
\[
\lim_{t \to +\infty} (\lambda t^q - F(t)) = +\infty
\]
and that the function \( t \mapsto \lambda t^q - F(t) \) has a unique global minimum in \([0, +\infty[ \). Finally, suppose that condition \((j_0^+)\) holds with \( \liminf_{t \to 0^+} F(t)/t^p = 0 \).

Under such hypotheses, problem \((\mathcal{P})\) has a sequence of non-zero non-negative weak solutions \( \{u_n\} \) with
\[
\lim_{n \to \infty} ||u_n|| = 0.
\]

Proof. We apply Theorem 3.1 to the functions
\[
f_*(t) = \begin{cases} f(0) & \text{if } t \leq 0, \\ f(t) & \text{if } t > 0, \end{cases}
\]
\[
\psi_*(t) = |t|^q.
\]
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It is clear that $\tilde{f} \in \mathcal{A}$ and $\psi \in \mathcal{F}_q$. Since

$$\tilde{F}(t) = \int_0^t \tilde{f}(s) \, ds = \begin{cases} f(0)t & \text{if } t \leq 0, \\ F(t) & \text{if } t > 0, \end{cases}$$

we obtain the coercivity of the function $\lambda \psi - \tilde{F}$ for every $\lambda > 0$. Also, as $f(0) \geq 0$,

$$\lambda \psi(t) - \tilde{F}(t) = \lambda |t|^q - f(0)t > 0$$

for every $t < 0$. From the assumptions there exists a unique $t_\lambda \geq 0$ such that

$$\lambda \psi(t_\lambda) - F(t_\lambda) \leq \lambda \psi(t) - F(t)$$

for every $t > 0$. Therefore,

$$\lambda \psi(t_\lambda) - \tilde{F}(t_\lambda) \leq 0 < \lambda \psi(t) - \tilde{F}(t)$$

for every $t < 0$, which implies the uniqueness of the global minimum of the function $\lambda \psi - \tilde{F}$ in $\mathbb{R}$.

In this framework, given that $a = 0$ and $b = +\infty$, it follows that

$$\alpha(\tilde{F}, \psi, b) = 0 \quad \text{and} \quad \beta(\tilde{F}, \psi, a) > 0.$$

From our assumptions, the existence of a sequence $\{t_n\}$ with $t_n \to 0^+$ such that

$$\lim_{n \to \infty} \frac{F(t_n)}{t_n^p} \to 0$$

also follows. Put $r_n = \psi(t_n) = t_n^q$. It is clear that $r_n \to 0^+$ and that $\psi^{-1}(r_n) = \{t_n, -t_n\}$. Therefore,

$$\limsup_{n \to \infty} \sup_{r_n^{p/q}} \tilde{F} = \limsup_{n \to \infty} \sup_{r_n^{p/q}} \left\{ \frac{F(t_n) - f(0)t_n}{t_n^p} \right\} \leq \lim_{n \to \infty} \frac{F(t_n)}{t_n^p} = 0,$$

which implies condition (3.1) for $\tilde{F}$. All the assumptions of Theorem 3.1 are fulfilled, so the functional $\tilde{\mathcal{E}}: W^{1,p}_0(\Omega) \to \mathbb{R}$,

$$\tilde{\mathcal{E}}(u) = \frac{1}{p} \|u\|^p - \int_{\Omega} h(x) \tilde{F}(u(x)) \, dx,$$

has a sequence of non-zero critical points $\{u_n\}$ with $\lim_{n \to \infty} \|u_n\| = 0$. In order to conclude the proof it is enough to show that each critical point of $\tilde{\mathcal{E}}$ is non-negative, that is, it is a solution of $(\mathcal{P})$. Indeed, if $u$ is a critical point of $\tilde{\mathcal{E}}$, then

$$\int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) \, dx = \int_{\Omega} h(x) \tilde{f}(u(x)) v(x) \, dx$$

for every $v \in W^{1,p}_0(\Omega)$, and by choosing as test function $v = -u^- = \min\{0, u\}$ we obtain

$$\|u^-\|^p = \int_{\{u \leq 0\}} h(x) f(0) u(x) \, dx \leq 0,$$

that is, $u \geq 0$ a.e. in $\Omega$. The claim follows. \qed
3.2. The $\infty$-case

The existence of infinitely many solutions at infinity is possible by ensuring that $\beta = +\infty$.

**Theorem 3.8.** Let $f \in A$ and let $h \in L^\infty(\Omega) \setminus \{0\}$ with $h \geq 0$. Assume that there exists $\psi \in F_q$ such that, for each $\lambda \in ]a, b[$, the function $\lambda \psi - F$ is coercive and has a unique global minimum in $\mathbb{R}$. Finally, suppose that

$$\alpha(F, \psi, b) < +\infty \quad \text{and} \quad \beta(F, \psi, a) = +\infty,$$

where

$$\lim_{r \to +\infty} \sup_{\psi^{-1}(r) \setminus \{0\}} \frac{F}{r^{p/q}} < \frac{1}{p(\gamma \psi \text{ess sup}_\Omega h c_q)^{p/q}(\int_\Omega h(x)\,dx)^{(q-p)/q}},$$

and $\mathcal{E}$ is unbounded from below.

Under such hypotheses, problem $(\mathcal{P})$ has a sequence of weak solutions $\{u_n\}$ with

$$\lim_{n \to \infty} \|u_n\| = +\infty.$$

Also, $\mathcal{E}(u_n) < 0$ for any $n \in \mathbb{N}$ and $\{\mathcal{E}(u_n)\}$ is decreasing.

**Proof.** From (3.6) we can fix an increasing sequence $\{r_n\}$ of positive numbers such that $\lim_n r_n = +\infty$ and

$$\sup_{\psi^{-1}(r_n)} F < \frac{r_n^{p/q}}{p(\gamma \psi \text{ess sup}_\Omega h c_q)^{p/q}(\int_\Omega h(x)\,dx)^{(q-p)/q}} \quad \forall n \in \mathbb{N}. \quad (3.7)$$

Without loss of generality we can assume that $\alpha(F, \psi, b) < r_n < \beta(F, \psi, a) = +\infty$ for every $n \in \mathbb{N}$. From Theorem 2.1 we deduce that for every $n \in \mathbb{N}$ there exists a solution of $(\mathcal{P})$, $u_n \in W^{1,p}_0(\Omega)$, with

$$\|u_n\|^p < dr_n^{p/q}.$$

In particular, from Theorem 2.1, $u_n$ is a global minimum of the restriction of $\mathcal{E}$ to $B(0, dr_n^{p/q})$. We claim that $\{u_n\}$ is unbounded. Assume by contradiction that there exists $M > 0$ such that

$$\|u_n\| \leq M \quad \text{for every } n \in \mathbb{N}.$$

Then, by the reflexivity of $W^{1,p}_0(\Omega)$, there exists a subsequence $\{u_{n_k}\}$ and $\tilde{u} \in W^{1,p}_0(\Omega)$ such that

$$u_{n_k} \rightharpoonup \tilde{u} \quad \text{in } W^{1,p}_0(\Omega).$$

Let $u \in W^{1,p}_0(\Omega)$. Since $r_n \to +\infty$, there exists $\nu \in \mathbb{N}$ such that $u \in B(0, dr_{\nu k}^{p/q})$ for $k \geq \nu$. So,

$$\mathcal{E}(\tilde{u}) \leq \liminf_{k \to \infty} \mathcal{E}(u_{n_k}) \leq \mathcal{E}(u),$$

which means that $\tilde{u}$ is a global minimum for $\mathcal{E}$, a contradiction. The thesis follows. \hfill $\square$

**Remark 3.9.** It is clear that Theorem 3.8 holds with (3.7) in place of (3.6).

Notice that it is crucial to require that $\mathcal{E}$ has no global minima.
**Lemma 3.10.** Assume that one of the following conditions holds:

(i) $-\infty < \liminf_{t \to +\infty} \frac{F(t)}{t^p} < \limsup_{t \to +\infty} \frac{F(t)}{t^p} = +\infty$,

(ii) $-\infty < \liminf_{t \to -\infty} \frac{F(t)}{|t|^p} < \limsup_{t \to -\infty} \frac{F(t)}{|t|^p} = +\infty$,

(k) $\text{ess inf}_{\Omega} h > 0$ and $\liminf_{t \to +\infty} \frac{F(t)}{t^p} > \frac{1}{pc_p \text{ess inf}_{\Omega} h}$,

(h) $\text{ess inf}_{\Omega} h > 0$ and $\liminf_{t \to -\infty} \frac{F(t)}{|t|^p} > \frac{1}{pc_p \text{ess inf}_{\Omega} h}$.

Then $\mathcal{E}$ is unbounded from below.

In the same spirit as Remark 3.4, we have the following remark.

**Remark 3.11.** Condition (i) holds if

(j) $\liminf_{t \to +\infty} \frac{F(t)}{t^p} > -\infty$ and $\limsup_{t \to +\infty} \frac{F(t)}{t^p} > (2^N - 1) \left[ p \liminf_{t \to +\infty} \frac{F(t)}{t^p} + \frac{2^p \|h\|_{\infty} \theta^p}{\|h\|_{\infty} \theta^p} \right]$.

Analogously, for (i) we need

(j) $\liminf_{t \to -\infty} \frac{F(t)}{t^p} > -\infty$ and $\limsup_{t \to -\infty} \frac{F(t)}{t^p} > (2^N - 1) \left[ p \liminf_{t \to -\infty} \frac{F(t)}{t^p} + \frac{2^p \|h\|_{\infty} \theta^p}{\|h\|_{\infty} \theta^p} \right]$.

**Remark 3.12.** Notice that actually the unboundedness of $\mathcal{E}$ follows from the weaker assumptions

$-\infty < \liminf_{t \to +\infty} \frac{F(t)}{t^p} \leq \limsup_{t \to +\infty} \frac{F(t)}{t^p} = +\infty$

and

$-\infty < \liminf_{t \to -\infty} \frac{F(t)}{|t|^p} \leq \limsup_{t \to -\infty} \frac{F(t)}{|t|^p} = +\infty$.

It is worth pointing out that assumption (i) (respectively, (i)) cannot be replaced by the condition

$\lim_{t \to +\infty} \frac{F(t)}{t^p} = +\infty$ (respectively, $\lim_{t \to -\infty} \frac{F(t)}{|t|^p} = +\infty$).

Indeed, if this was the case, then

$\lim_{t \to +\infty} F(t) = +\infty$.

From the coercivity of $\lambda \psi - F$, one also has that $\lim_{t \to +\infty} \psi(t) = +\infty$. From Remark 2.2, for every $r \in [\alpha, +\infty]$ there exists $t_r \in \psi^{-1}(r)$, and so $t_r \to +\infty$ as $r \to +\infty$.

As $\psi \in \mathcal{F}_q$, there exists a positive number $c$ such that

$\psi(t) \leq c|t|^q$ for any $t$. 

Then, in particular, 
\[ \psi(t_r)^{p/q} \leq c'|t_r|^p \] (notice that \( \psi \) and \( F \) are definitively positive). So
\[
\sup_{r \in \Omega} \frac{F(t_r)}{\psi(t_r)^{p/q}} \geq \frac{F(t_r)}{\psi(t_r)^{p/q}} \geq \frac{F(t_r)}{c'^p} \]
and the latter goes to \(+\infty\). So, (3.6) is contradicted.

**Question 3.13.** Which kind of assumptions on \( f \) ensure that \( \mathcal{E} \) has no global minimum but is still bounded from below?

We conclude the section with the following counterpart of Theorem 3.7.

**Theorem 3.14.** Let \( f \in A \) with \( f(0) \geq 0 \), let \( h \in L^\infty(\Omega) \setminus \{0\} \) with \( h \geq 0 \). Assume that, for each \( \lambda \in [0, +\infty[ \),
\[
\lim_{t \to +\infty} (\lambda t^q - F(t)) = +\infty
\]
and the function \( t \mapsto \lambda t^q - F(t) \) has a unique global minimum in \([0, +\infty[\). Finally, suppose that condition (j+\infty) holds with \( \lim \inf_{t \to +\infty} F(t)/t^p = 0 \).

Under such hypotheses, problem (\( \mathcal{P} \)) has a sequence of non-negative weak solutions \( \{u_n\} \) with
\[
\lim_{n \to \infty} \|u_n\| = +\infty.
\]

4. Applications

In this section we provide some examples and applications of the theory. In particular, we want to stress the novelty of our results.

As mentioned in the introduction, our results can be applied when
\[
\lim_{|t| \to \infty} \frac{F(t)}{|t|^p} \in \mathbb{R},
\]
which represents a novelty compared with the many papers existing on the subject (see, for example, [10, 13, 15]).

**Example 4.1.** Let \( h(x) \equiv 1 \), let \( |\Omega| = 1 \) and let
\[
F(t) = \frac{1}{p c_p} |t|^p - G(t),
\]
where \( G : \mathbb{R} \to \mathbb{R} \) is a \( C^1 \) function that will be specified later. Choose \( \psi(t) = |t|^p \) and put
\[
h_\lambda(t) = \lambda \psi(t) - F(t) = \lambda |t|^p - \frac{1}{p c_p} |t|^p + G(t) = \left( \lambda - \frac{1}{p c_p} \right) |t|^p + G(t).
\]
In order to apply Theorem 3.8 we need the coercivity of \( h_\lambda \) for \( \lambda \in [1/p c_p, +\infty[ \) and the uniqueness of the global minimum of \( h_\lambda \) for \( \lambda \in [1/p c_p, +\infty[ \).

Equivalently, we require that
\[
(g_1) \text{ the function } t \mapsto \tau |t|^p + G(t) \text{ is coercive for } \tau > 0,
\]
\[
(g_2) \text{ the function } t \mapsto |t|^p + G(t) \text{ has a unique global minimum in } \mathbb{R} \text{ for } \tau > 0.
\]
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We will also need that

\((g_3)\) \(G(0) = 0, G\) has no global minima and \(\lim_{|t| \to \infty} \frac{G(t)}{|t|^p} = 0;\)

\((g_4)\) there exists a divergent sequence \(\{r_k\}\) such that \(\min\{G(r_k), G(-r_k)\} > 0\) for all \(k;\)

\((g_5)\) the energy

\[\mathcal{E}(u) = \frac{1}{p} \|u\|^p - \frac{1}{pc_p} \|u\|^p + \int_{\Omega} G(u) \, dx\]

is unbounded from below.

If the above conditions hold, then \(\alpha = 0, \beta = +\infty\) and (3.7) holds (see Remark 3.9).

Let us propose a concrete example of a function \(G\) fulfilling the above assumptions: choose \(p = 2, N = 1\) and

\[G(t) = \begin{cases} |t|^{3/2} \sin(\ln(t^2) - 3/4), & t \leq 0, \\ |t|^{3/2}, & t > 0. \end{cases}\]

In order to verify \((g_5)\) it is enough to test \(\mathcal{E}\) on the sequence \(v_k = \mu_k u_1\) \((u_1\) being the first eigenfunction of \(-\Delta_p\)) and \(\mu_k = -e^{2k\pi} \to -\infty.\) Since \(\int_{0}^{1} G(u_1(x)) \, dx < 0,\) we have \(\mathcal{E}(v_k) \to -\infty.\) Theorem 3.8 applies.

The last example emphasises the applicability of the present theory (Theorems 3.7 and 3.14) to a higher-dimensional framework (i.e. when \(1 < p < N\)).

**Example 4.2.** Let \(p = 2, N = 3\) and \(q = 5,\) and let \(h\) and \(\theta\) be as in the general framework. For \(\eta > 0\) to be chosen later, define \(F: [0, +\infty[\) by putting (see Figure 3)

\[F(t) = \eta t^2(\sin(\ln t^2) + 1).\]
We extend $F$ to the whole real line by putting $F(t) = 0$ for every $t \leq 0$. Then
\[ f(t) = F'(t) = 2\eta t [\sin(\ln t^2) + \cos(\ln t^2) + 1] \]
for every $t > 0$ and $f(t) = 0$ for every $t \leq 0$. Clearly, $f \in \mathcal{A}$ and for every $\lambda > 0$
\[ \lim_{t \to +\infty} (\lambda t^5 - F(t)) = +\infty. \]
Also, for every $\lambda > 0$ it is possible to see (see Figure 4) that the function $t \to \lambda t^5 - F(t)$
has a unique global minimum in $[0, +\infty[$. We also have that
\[ \liminf_{s \to 0^+} \frac{F(s)}{s^2} = 0 \quad \text{and} \quad \limsup_{s \to 0^+} \frac{F(s)}{s^2} = 2\eta, \]
and
\[ \liminf_{s \to +\infty} \frac{F(s)}{s^2} = 0 \quad \text{and} \quad \limsup_{s \to +\infty} \frac{F(s)}{s^2} = 2\eta. \]
Theorems 3.7 and 3.14 are applicable if we choose
\[ \eta > \frac{14}{\|h\|_\infty^2}. \]

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